Ecole Doctorale IAEM Lorraine

THESIS

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By

Francisco Ronay LÓPEZ ESTRADA

Model-Based Fault Diagnosis Observer Design for Descriptor LPV System with Unmeasurable Gain Scheduling.

Rapporteurs :	Olivier SENAME	Full Professor at the Université de Grenoble-INP
	Mohammed CHADLI	Associate Professor, HDR at the Université de Picardie Jules Verne
Reviewers :	Mickael RODRIGUES	Associate Professor the Université de Lyon 1
	Flor Lizeth TORRES ORTÍZ	Associate Professor at the Universidad Nacional Autónoma de México, Mexico
	Didie MAQUIN	Full Professor at the Université de Lorraine
Thesis Directors :	Didier THEILLIOL	Full Professor at the Université de Lorraine
	Jean Christophe PONSART	Associate Professor, HDR at the Université de Lorraine
	Carlos Manuel ASTORGA ZARAGOZA	Professor at the Centro Nacional de Investigación y Desarrollo Tecnológico (CENIDET), Mexico

Centre de Recherche en Automatique de Nancy

CRAN – CNRS UMR 7039

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THÈSE

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Docteur de l'Université de Lorraine

(Spécialité Automatique, Traitement du Signal et Génie Informatique)

par

Francisco Ronay LÓPEZ ESTRADA

Contribution au diagnostic de défauts à base de modèles : Synthèse d'observateurs pour les systèmes singuliers linéaires à paramètres variants aux fonctions d'ordonnancement non mesurables.

Rapporteurs :	Olivier SENAME	Professeur des Universités, Grenoble-INP
	Mohammed CHADLI	Maître de Conférences, HDR à l'Université de Picardie Jules Verne
Examinateurs :	Mickael RODRIGUES	Maître de Conférences à l'Université de Lyon 1
	Flor Lizeth TORRES ORTÍZ	Associate Professor à l' Universidad Nacional Autónoma de México, Mexique
	Didier MAQUIN	Professeur des Universités, Universié de Lorraine
Directeurs de thèse :	Didier THEILLIOL	Professeur des Universités, Université de Lorraine
	Jean Christophe PONSART	Maître de Conférences, HDR à l'Université de Lorraine
	Carlos Manuel ASTORGA ZARAGOZA	Professeur au Centro Nacional de Investigación y Desarrollo Tecnológico (CENIDET), Mexique

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Publications

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- **F. R. LÓPEZ-ESTRADA**, D. THEILLIOL, C.M. ASTORGA-ZARAGOZA, and J. C. PONSART, (2012). Developments of a Scilab/Matlab toolbox dedicated to LTI/LPV descriptor systems for fault diagnosis. 10th European Workshop on Advanced Control and Diagnosis, ACD 2012, Copenhagen, Denmark.
- **F. R. LÓPEZ-ESTRADA,** J. C. PONSART, C. ASTORGA-ZARAGOZA and D. THEILLIOL, (2013). Fault estimation observer design for descriptor-LPV systems with unmeasurable gain scheduling functions. 2nd IEEE International Conference on Control and Fault-Tolerant Systems (SYSTOL), Nice, France.
- **F. R. LÓPEZ-ESTRADA**, J. C. PONSART, D. THEILLIOL, C. M. ASTORGA-ZARAGOZA, and S. ABERKANE (2014). Fault Diagnosis Based on Robust Observer for Descriptor-LPV Systems with Unmeasurable Scheduling Parameters. In The 19th World Congress of the International Federation of Automatic Control. Cape Town, South Africa.
- **F. R. LÓPEZ-ESTRADA**, J. C. PONSART, D. THEILLIOL, and C. M. ASTORGA-ZARAGOZA. Robust Sensor FDI Observer for Polytopic Descriptor LPV systems with Unmeasurable Scheduling Varying Parameter. In American Control Conference. Portland, OR, USA.
- **F. R. LÓPEZ-ESTRADA**, J. C. PONSART, D. THEILLIOL, C. M. ASTORGA-ZARAGOZA, and Y. ZHANG, (2014). Robust Sensor Fault Diagnosis and Tracking Controller for a UAV Modelled as LPV System. IEEE International Conference on Unmanned Aircraft Systems. Orlando, Florida, USA.
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Software development

Additionally, the following toolbox is continuously updated and maintained.

• **F. R. LÓPEZ-ESTRADA**, J-C Ponsart, C. Astorga-Zaragoza, and D. Theilliol. Descriptor LTI and LPV Systems Calculation Tool Kit. Matlab version: http://www.mathworks.fr/matlabcentral/fileexchange/40589-descritor-lti-and-lpv-calculation-tool-kit-v-1-21 Scilab version: downloaded from the ATOMS plataform with more of 2600 downloads.

Résumé

Ce mémoire de thèse est consacré à la conception de méthodes de diagnostic à base de modèles fondées sur les observateurs pour les systèmes non linéaires modélisés comme des systèmes singuliers (D) linéaires à paramètres variants (LPV), notés D-LPV (Descriptor-Linear Parameter Varying). Les systèmes D-LPV constituent une classe particulière de systèmes approximant avec un certain degré de précision la dynamique des systèmes complexes non linéaires à partir d'une combinaison de modèles linéaires locaux pondérés par des fonctions convexes d'ordonnancement. Dans le contexte de l'apparition de défauts capteurs ou actionneurs, ce travail de thèse s'attache aux systèmes pour lesquels ces fonctions d'ordonnancement sont non mesurables mais dépendent de l'état du système. Afin de détecter et isoler des défauts, ce travail de thèse développe des synthèses d'observateurs appropriés en développant des nouvelles conditions suffisantes en termes d'inégalités matricielles linéaires (LMI) pour garantir la synthèse de résidus sensibles aux défauts et robustes aux erreurs d'estimation inhérentes aux fonctions d'ordonnancement non mesurables.

- Etendant des méthodes H_∞ afin d'effectuer l'estimation d'état, la détection de pannes, la localisation et la reconstruction de défaut sur les capteurs ;
- Garantissant une sensibilité “optimale” aux pannes vis-à-vis du rejet de perturbations au travers le développement d'observateurs de type H_-/H_∞.

À cette fin, le mémoire de thèse est organisé en cinq chapitres :

Le Chapitre 1 est consacré à l'introduction générale, aux objectifs et contributions de ce travail.

Le Chapitre 2 présente les éléments nécessaires pour décrire la représentation, la modélisation, les propriétés, l'analyse et la conception d'observateur pour les systèmes D-LPV ainsi qu'un état de l'art détaillé des travaux associés à ce thème de recherche.

Le Chapitre 3 est dédié au développement de trois méthodes différentes fondées sur la théorie H_∞ pour concevoir des observateurs de détection de défaut pour les systèmes D-LPV. Les méthodes proposées sont appliquées à un exemple dans le cadre de la détection de défaut capteurs. L'isolation de ces défauts est mise en œuvre au travers un banc d'observateurs et les performances de chacune des trois méthodes sont comparées.

Le Chapitre 4 propose une méthode de détection de défauts sur la base d'observateurs établis sur le principe H_-/H_∞, tenant compte ainsi d'un meilleur compromis entre la sensibilité aux pannes et la robustesse aux perturbations. De nouvelles conditions suffisantes à l'aide de LMI sont proposées afin de résoudre le problème de synthèse du gain des observateurs.

Le dernier chapitre est dédié à la conclusion générale et à l'analyse de problèmes ouverts pouvant être abordés dans des travaux futurs.

Abstract

This work is dedicated to the synthesis of model-based fault detection and isolation (FDI) techniques based on observers for nonlinear systems modeled as Descriptor-Linear Parameter Varying (D-LPV) systems. D-LPV systems are a particular class of systems that can represent (or approximate in some degree of accuracy), complex nonlinear systems by a set of linear local models blended through convex parameter-dependent scheduling functions. The global D-LPV System can describe both time-varying and nonlinear behavior. Nevertheless, in many applications the time-varying parameters in the scheduling functions could be unmeasurable. Models which depend on unmeasurable scheduling functions cover a wide class of nonlinear systems compared to models with measurable scheduling functions, but the design of control schemes for D-LPV systems with unmeasurable scheduling functions are more difficult than those with a measurable one, because the design of such control schemes involve the estimation of the scheduling vector. This topic is addressed in this work by considering the following main targets:

- to design FDI in D-LPV systems based on H_∞ observers in order to guarantee robustness against disturbances and errors due the unmeasurable gain scheduling functions
- to extend the proposed H_∞ methods to perform state estimation and fault detection, isolation and fault magnitude estimation in the case of sensor faults
- to guarantee the best trade-off between fault sensitivity and disturbance rejection by developing H_-/H_∞ fault detection observers for D-LPV systems.

The thesis is organized as follows

Chapter 1 is dedicated to provide a general introduction, the objectives and contribution of this work.

Chapter 2 is organized in order to provide the minimum necessary elements to describe the representation, modeling, properties, analysis, and observer design of D-LPV systems. Chapter 2 is also dedicated to a detailed review of the state of the art.

Chapter 3 is dedicated to the development of three different methods to design fault detection observers for D-LPV systems based on H_∞ theory. Finally, the proposed methods are applied to an example, for sensor fault detection and isolation by means of an observer bank, in order to compare the performance of each method.

Chapter 4 is dedicated to the design of a FDI method based on observers with $H_-\ /H_\infty$ performance. Based on the H_-/H_∞ approach, which considers the best trade-off between fault sensitivity and robustness to disturbance, adequate LMIs are obtained to guarantee sufficient conditions for the design problem. In order to illustrate the effectiveness of the proposed techniques, an example is considered.

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Notations and Acronyms

Matrices and vectors

$*$ denotes the transposed element in the symmetric positions of a matrix

$\hat{\rho}_i$ estimated gain scheduling function of the ith model

$\mathbb{C}$ set of complex numbers

$\mathbb{R}$ set of real numbers

$\mathbb{R}^+$ set of positive real numbers

A the rank of a matrix A

$\mathsf{He}\{A\}$ shorthand notation for $A + A^T$

$\| G \|_-$ H_- index of G

$\| G \|_\infty$ H_∞ norm of G

ρ_i unmeasurable gain scheduling function of the ith model

φ, υ vector functions of appropriate dimensions

$\zeta(t)$ Scheduling vector

A^{-1} the inverse of a matrix A

$A^\dagger$ the pseudo-inverse of a matrix A

A^T the transpose of a matrix A

n_ι Number of finite modes in a descriptor system

$P < 0, P \leq 0$ matrix P is negative definite, negative semidefinite

$P > 0, P \geq 0$ matrix P is positive definite, positive semidefinite

$r = \mathsf{E}$ Number of dynamic modes

Acronyms

D-LPV Descriptor-Linear Parameter Varying

D-LTI Descriptor-Linear Time Invariant

DAE Differential Algebraic Equation

FDO fault detection observer

LPV Linear Parameter Varying

LTI Linear Time Invariant

LMI Linear Matrix Inequality

Chapter 1

Introduction

Contents

1.1 Context of the Thesis

The results presented in this report are the fruit of three years of work. The research is developed under a *cotutelle* agreement between the National Center of Research and Technological Development (CENIDET), Cuernavaca, Mexico, and the University of Lorraine, Nancy, France. The research was jointly supervised by Carlos Manuel ASTORGA ZARAGOZA (Professor at CENIDET), Jean Christophe PONSART, and Didier THEILLIOL (respectively associate professor HdR and Professor). The research was sponsored by the National Council for Science and Technology (CONACyT, Mexico) and the Ministry of Foreign Affairs and International Development (with the *EIFFEL* scholarship during 10 months), France; the supports are gratefully acknowledged.

1.2 Introduction

The ability to detect and isolate faults in complex systems is necessary to ensure their effectiveness, fulfill dependability requirements, and improve their safety. Generally, a fault is something that changes the behavior of a system, such that it does no longer satisfy its purpose (Lunze et al., 2006). To illustrate this concept, we can consider a system which output $y(t)$ is manipulated by the input $u(t)$. In the fault-free operation, the input/output relationship holds. However, when the system is affected by a fault, the relationship changes because the output is affected by both the input and the unknown fault. This unknown fault can be related to component failures, actuator failures, or instrument failures in the sensors. For example, the darkest moment in the French aviation is related to the crash of the Air France flight 447 (from Rio de Janeiro to Paris) that caused the death of its 228 passengers. The final report concluded that the aircraft crashed after temporary inconsistencies between the airspeed measurements, due to the pitot tubes being obstructed by ice crystals, which caused the disconnection of the autopilot that eventually lead to an aerodynamic stall, from which the aircraft did not recover and crashed (BEA, 2012). This example shows the importance of developing fault detection and isolation (FDI) systems to avoid systems breakdowns, failures or disasters.

In the literature, this problem has been extensively addressed by considering different approaches, e.g. parity space (Gertler, 1997), identification (Isermann, 1984; Torres et al., 2012), fault detection filters (Chen and Patton, 1999; Wang et al., 2007; Ding, 2008), among others. Among the available FDI methods, the observer-based ones have become one of the most successful techniques. The observer-based FDI method can be regarded as a three step process. First, some virtual outputs are generated by the observer and compared with the measurement outputs of the real process. The difference between the analytical and the measured outputs generate error signals named "residuals". Second, the residuals are tested under predefined threshold to generate a particular signature. Finally, the signature is analyzed in order to obtain some information about the presence or not of the considered faults (fault detection). It could be also possible to isolate the faulty element (fault isolation) (Frank, 1996). Moreover, most of the proposed methods focus on designing FDI systems for linear-time invariant (LTI) systems. Nevertheless, it is well known that all systems exhibit a nonlinear behavior. In general, the real design of a nonlinear fault detection method is not an easy task, even if the nonlinear system is completely known (Alcorta-García et al., 2013). Indeed, in order to obtain more reliable residuals, the quality of the model plays an important role for both fault detection and isolation.

More recently, linear parameter varying (LPV) systems have gained much interest because they can represent nonlinear systems via a finite number of linear time invariant (LTI) models. Generally, the LTI models, which compose the LPV systems, are blended through on-line measurable scheduling functions which depend on the input, the output of the system, or an external parameter. In the literature, there is a wide variety of design methods and tools for LPV systems (see, e.g., the survey papers (Xiao and Gao, 2007; Lopes-dos Santos et al., 2012; Hoffmann and Werner, 2014), the references therein and the recent books (Mohammadpour and Scherer, 2012; Sename et al., 2013a; Briat, 2014)). On the other hand, descriptor systems, also referred as singular systems, implicit systems, or differential-algebraic systems, are mathematical models with the property of integrating both static (algebraic) and dynamical (differential) equations in the same model. Generally, descriptor systems give a more complete class of dynamical models than the standard state-space representation. In contrast, designing a FDI method for descriptor systems is more difficult than for regular systems since they usually have three types of modes: finite dynamic modes, impulsive modes and non-dynamic modes (Dai, 1989; Duan, 2010). Furthermore, it is logical that the integration of the descriptor and the LPV theory, also known as descriptor-LPV (D-LPV) systems, can improve the capacity of describing a large class of physical systems. A detailed background about D-LPV system will be addressed in Chapter 2.

As discussed above, both LPV and D-LPV systems are blended by scheduling functions. However, in many applications, the scheduling functions depend on the states, e.g. systems modeled by considering the nonlinear sector approach. In such cases, the states could be not measured. Unfortunately, the developed methods for systems with measurable scheduling functions are not directly applicable for systems with unmeasurable scheduling functions, and usually, a suitable design involves an observer to estimate the unmeasurable vector. To the best of the author's knowledge, the problem of fault diagnosis via state-observers for D-LPV systems with *unmeasurable scheduling functions* has not been fully investigated so far, and still remains an open and unsolved issue. This topic will be addressed in this thesis.

This thesis is dedicated to design robust state observers for systems modeled as D-LPV systems with unmeasurable gain scheduling functions. The thesis is organized in two parts. The first part of this thesis is dedicated to propose three different methods to design the observers. To guarantee robustness against disturbances, noise and errors given by the unmeasurable scheduling functions, the observer design is addressed by considering the H_∞ performance. As a result, new sufficient conditions are given in terms of Linear Matrix Inequalities (LMIs), which guarantee asymptotic convergence to zero of the estimation error, and generate residuals to detect and isolate sensor faults. In addition, one of the proposed H_∞ method is extended to perform fault diagnosis in the case of sensor faults. The second part of the thesis, is dedicated to design a H_-/H_∞ fault detection observer to guarantee the best trade-off between fault sensitivity and disturbance rejection. The contributions of each chapter are summarized below.

1.3 Main contributions

The H_∞ and H_-/H_∞ frameworks for linear time invariant systems have been extensively discussed in the literature (Van der Schaft, 1992; Wang et al., 2007; Wang and Yang, 2013). Nevertheless, for the case of LPV and D-LPV systems there are many open problems concerning to the design of state observers and fault diagnosis observers. Moreover, the works reported in the literature consider mainly the measurable scheduling case, which reduces the applicability of the proposed methods. The main contributions of each thesis chapter can be summarized as follows:

Chapter 2 is organized to provide the minimum necessary technical background to describe the representation, modeling, properties, analysis, and observer design of D-LPV systems.

Chapter 3 is dedicated to the development of three methods to design state observers for D-LPV systems. Three methodologies to design robust H_∞ observers for D-LPV systems with unmeasurable scheduling functions are proposed. The main target of these methods is to guarantee the best robustness against the disturbances and the error generated by the unmeasurable gain scheduling functions and then perform fault detection. For all various approaches, sufficient conditions to guarantee asymptotic convergence to zero of the observation error and residuals are obtained in terms of linear matrix inequalities (LMIs). Simulation examples are proposed to illustrate and compare the performance of each method. In addition, based on one of the proposed H_∞ observer, a state-estimation, fault detection, isolation and sensor fault reconstruction observer is proposed. Sufficient conditions are proposed to guarantee asymptotic convergence of the estimation error and effective fault estimation. Compared with the previous methods, this approach can reconstruct faults, even if they affect the system simultaneously.

Chapter 4 is dedicated to the design of a fault detection and isolation method based on a robust observer with H_-/H_∞ performance. Unlike the works published recently for discrete D-LPV systems (Boulkroune et al., 2013; Chadli et al., 2013a), which consider unmeasurable scheduling functions, the problem for continuous D-LPV system still remains open and unsolved. The contribution of this chapter is to propose sufficient conditions in the LMI framework to guarantee the best trade-off between disturbances and uncertainties attenuation and fault sensitivity on the generated residuals. A numerical example is presented to illustrate the effectiveness of the proposed method.

Chapter 5 is dedicated to provide a general conclusion of the research and to analysis some open problems that could be addressed in future works.

Appendix Three Appendices are included. The **Appendix A** is dedicated to provide complementary information about descriptor LTI systems. This includes information about different methods to computed observability, controllability, stability, among others. This information can be used as a complement of Chapter 3. **Appendix B** provides a catalog of important definitions and theorems extensively used in the literature concerning to LMIs. **Appendix C** presents a overview of some technical results concerning to tracking controllers and discrete-time systems.

Chapter 2

Overview of D-LPV systems

Contents

This Chapter is dedicated to the state of the art on Descriptor-Linear Parameter Varying (D-LPV) system. Mathematical preliminaries and theory are presented in this Chapter in order conceive the conceptual developments in the subsequent chapters. Section 2.1 is dedicated to developing representations of D-LPV systems and some important theoretical concepts are introduced. Section 2.2 is dedicated to the bibliographical review of relevant contributions devoted to regular LPV and D-LPV systems. In Section 2.3, two methods developed in the literature to obtain a D-LPV model from a nonlinear model are presented. In addition, three different D-LPV models of real systems not reported in the literature, are obtained in order to illustrate the effectiveness of the modeling methods. In Section 2.4, the basic assessing properties of D-LPV systems are introduced focusing on stability and observability, which are necessary to perform state observer design. Section 2.5 is dedicated to present some mathematical concepts developed to design D-LPV state observers, with the aim of illustrating the different problems presented in systems with measurable and unmeasurable gain scheduling function. Section 2.6 presents some computational tools developed as a result of the study of the state of the art. Finally, some conclusions are presented in Section 2.7.

2.1 Definition of D-LPV systems

Nonlinear systems are often modeled as systems of ordinary differential equations (ODEs) in order to represent the dynamics of the physical elements which describe the plants. Their state space description have the following general form

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1}$$

Nonetheless, there are several situations where some systems involve dynamic and constraints among the variables (for instance see (Brenan et al., 1995)). In such cases, a system of differential algebraic equations (DAEs) arise naturally. A DAE system reflects the properties of differential equations as well as the properties of algebraic equations. The general form of a DAE system is as follows:

$$\begin{cases} f(\dot{x}(t), x(t), u(t)) & = 0 \\ g(x(t), u(t), y(t)) & = 0 \end{cases} \tag{2.2}$$

where $x(t)$ is the state vector, $u(t)$ is the control input vector, $y(t)$ is the measured output vector. f and g are vector functions of $\dot{x}(t)$, $x(t)$, and $u(t)$, of appropriate dimensions. The descriptor representation of (2.2) can be written in the following form

$$\begin{cases} E\dot{x}(t) = \varphi(x(t), u(t)) \\ y(t) = \upsilon(x(t), \dot{x}(t)) \end{cases} \tag{2.3}$$

where φ and υ are functions in $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. The matrix E may be singular. Generally, the real design of nonlinear control techniques (control design, stability, fault detection methods, among others) is a hard task, even if the nonlinear system is completely known (Xu and Lam, 2006). Frequently, a descriptor linear time invariant (D-LTI) representation can be deduced to approximate the nonlinear system. If φ and υ are linear functions of $x(t)$ and $u(t)$, then the nonlinear system (2.3) is represented as

$$\begin{cases} E\dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{cases} \tag{2.4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state vector, the control input and the measured output vector respectively. E, A, B, C are constant matrices. Descriptor systems have many important applications, e.g. applications to circuit systems, robotics, neural networks (Newcomb and Dziurla, 1989), aircraft modeling (Masubuchi et al., 2004), complex systems (Nagy-Kiss et al., 2011). The following books propose many applications and also present | excellent reviews of descriptor linear system (Dai, 1989; Xu and Lam, 2006; Duan, 2010).

A D-LTI system (2.4) is defined around a linearization point, and therefore it is not possible to guarantee stability and performance for trajectories of the nonlinear system. An alternative to represent nonlinear systems is by considering the Linear Parameter Variant (LPV) approach. LPV systems have become very popular in the last years mainly because they can represent, or approximate with some degree of accuracy, complex nonlinear systems by a set of linear local models blended through convex scheduling functions. As a result of the scheduling functions, which imply parameter variations, the LPV system can describe both time-varying and nonlinear behavior. This property represents their main attractiveness, which consist in applying powerful techniques developed for D-LTI systems to nonlinear systems. Descriptor-LPV (D-LPV) systems are described in state-space form as

$$\begin{cases} E\dot{x}(t) & = \sum_{i=1}^{h} \rho_i(\zeta(t)) \left\{ A_i x(t) + B_i u(t) \right\} \\ y(t) & = Cx(t) \end{cases} \tag{2.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state vector, the control input and the measured output vector respectively. A_i, B_i, C are constant matrices of appropriate dimensions. It is assumed that C is a constant matrix, and therefore it is not affected by the scheduling functions that is a general assumption in the D-LPV framework, as considered in (Rodrigues et al., 2014; Nagy Kiss et al., 2011). The matrix E may be singular. h is the number of local models. $\rho_i(\zeta(t))$ represents the scheduling functions defined by the convex property

$$\begin{cases} \rho_i(\zeta(t)) \geq 0, \forall i \in [1, 2, ..., h], \\ \sum_{i=1}^{h} \rho_i(\zeta(t)) = 1, \forall t. \end{cases} \tag{2.6}$$

The scheduling functions depend on the scheduling vector $\zeta(t)$, that may contain the state, the input or some external parameter. Note that D-LPV systems are linear in $[x(t)^T \, u(t)^T]^T$ but nonlinear in the gain scheduling functions $\rho_i(\zeta(t))$.

Note that when the matrix E is an identity matrix, the system is called a standard normal LPV or regular LPV system, which has been intensely studied as detailed in the survey (Hoffmann and Werner, 2014). Nevertheless, this thesis is dedicated to descriptor systems as given in (2.5) with singular E matrix. In other words, we are interested in systems as given in (2.5), which dynamic order is defined by $r = \mathsf{rank}(E) < n$. Clearly, in the case of a regular LPV system, the dynamic order is n.

This research addresses the study of systems of the form given in (2.5) affected by disturbances. Additionally, it is considered that the scheduling functions depend on the unmeasurable state $x(t)$. More detailed descriptions will be done in the subsequent section and a detailed biographical review of systems with measurable and unmeasurable scheduling function is done below.

2.2 Biographical review

This section is dedicated to a biographical review of regular LPV and D-LPV systems. Both cases are considered in this review, because some ideas primarily developed for regular LPV system were extended to D-LPV systems. However, to the best of the author's knowledge, despite the existence of many works dedicated to regular LPV systems, only a few works are dedicated to D-LPV systems.

Note also that, despite in the past the gain scheduling functions were computed by considering Takagi-Sugeno fuzzy rules as defined in (Takagi and Sugeno, 1985), in the present Takagi-Sugeno (TS) and polytopic LPV systems are described by the same form. The community of people working on TS models uses the name “TS FUZZY systems”, even if with the recent modeling approaches (for example sector nonlinearity transformation), the obtained model is no longer “fuzzy” because the weighting functions are completely deterministic that corresponds to LPV or quasi-LPV systems as detailed in (Rodrigues et al., 2014). Therefore, in this review some works published as TS, which correspond to previous remark, are considered as LPV systems.

Considering the previous remarks, the biographical review is organized in two subsections. The first subsection is dedicated to systems with measurable scheduling functions and the second to systems with unmeasurable scheduling functions. This consideration is done because despite both have the same structure, the methodologies to analyze systems with unmeasurable scheduling functions are different compared to systems with measurable scheduling functions, as will be detailed in subsequent Sections. Also, each subsection is dedicated to first review works related to regular LPV systems and then those related to D-LPV systems, in order to illustrate the contrasts of both cases.

2.2.1 Measurable scheduling functions case

LPV systems were introduced by (Shamma and Athans, 1988, 1990) as mathematical models to design and to guarantee a suitable closed-loop performance for a given plant in different operating conditions, such that the scheduling parameter captured the nonlinearities of the plant. The term LPV was adopted to distinguish these systems from both linear time invariant (LTI) and linear time varying (LTV) systems. The distinction with respect to LTI systems is clear, because LPV systems are non-stationary. On the other hand, LPV systems are distinguished from LTV systems in the perspective taken on both analysis and synthesis. LPV system can be seen as a family of LTV systems, where each model is parameter varying according to the scheduling functions. Therefore, properties as stability, disturbances rejection, tracking, among others, hold for a family of LTV systems, rather than a single LTV system (Shamma, 2012).

In last years, significant progress has been made for LPV systems. For example, in the presence of uncertainties or disturbances, LPV robust ***control techniques*** have proved to have better performance than robust LTI controllers (Sato, 2011). Indeed, many solutions for LTI systems given in the Linear Matrix Inequalities (LMI) framework have been extended to LPV systems, e.g. a stabilization method for an arm-driven inverted pendulum was proposed in (Kajiwara et al., 1999); the proposed LPV controller was shown to outperform classical robust control techniques H_∞ and μ-synthesis. However, the method does not guarantee that the closed loop system exhibit the robust performance considered for the operation conditions. To handle this problem, a parametrized H_∞ was presented in the work of (Bruzelius et al., 2002); the control showed good performance when applied to a turbo fan jet engine. Other H_∞ controller for systems affected by time-varying parametric uncertainties can be consulted in (Scherer, 2004). In order to improve the performance of H_∞ controllers, a switching controller designed with multiple Lyapunov functions was proposed by (Xu et al., 2011b). Similarly, in (Xu et al., 2011a) an LPV control for switched systems with slow-varying parameters was proposed by adopting the blending method proposed by (Shin et al., 2002); the blending method separates the entire parameter set into overlapped subsets and an LPV controller for the whole region is blended by regional controllers. The resulting methodology was applied to an F-16 Aircraft Model. Nevertheless, the method is applicable under the assumption that the scheduling parameters can be measured on-line, which often is difficult to satisfy in practice. To solve this problem, a robust compensator designed for stable polytopic LPV plants, which considers prior and non-real-time knowledge of the dependent parameter, was proposed by (Xie et al., 2003). A bibliographical review of identification LPV methods for real application is given in (Giarre et al., 2006).

Recently, several ***Model Predictive Control*** (MPC) schemes have been proposed, e.g. (Lee et al., 2011) proposed an MPC method to stabilize LPV systems with delayed state; as a result, sufficient conditions which guarantee the asymptotic stability were obtained. A Quasi-min-max algorithm was developed in (Park et al., 2011); first, an off-line method for designing a robust state observer was obtained by considering LMI techniques then, an on-line optimization algorithm was applied to implement a robust stable controller with input constraints. Another difficulty in dealing with LPV plants is that the one-step ahead state prediction set is non-convex and depends quadratically on the scheduling vector. The problem has been solved by using the polytopic framework, which exploits the symmetrical property existing between any pairs of elements of the quadratic form. An improvement of these method can be consulted in (Casavola et al., 2012), where an ellipsoidal MPC strategy for discrete-time polytopic LPV systems subject to bounded disturbances, input and state constraints, was presented.

Indeed, LPV systems have proved to handle successfully ***real application problems***, some of them are mentioned as follows: an semi-active suspension controller was proposed in (Poussot-Vassal et al., 2008); the authors developed a LPV method in order to guarantee internal stability and some performance criteria for the semi-active suspension. (Abbas and Herbert, 2011) proposed a method for frequency-weighted discrete-time LPV model reduction with guaranteed stability of the reduced-order model, when both input and output weighting filters are used; the method was applied to a beam-head assembly of an industry-grade prototype gantry robot. A gain scheduling controller for the electronic throttle body in ride-by-wire racing motorcycles was proposed by (Corno et al., 2011), which considered model-based gain-scheduled position control system for throttle position tracking. An LPV system identification algorithm to model a leakage detection in high pressure natural gas transportation networks can be consulted in (Lopes et al., 2011). Many other successfully applications have been proposed, e.g. (Pfifer and Hecker, 2011) proposed an algorithm for modeling an industrial, highly nonlinear missile model, which allowed to prove the robust stability for a large region of the flight envelope. (Diaz-Salas et al., 2011) proposed a Magneto-Rheological (MR) LPV model that provided a control-oriented dynamical description suitable to design control strategies and to enhance comfort, steering, and road grid. Other MR model used to attenuate the vibration of a two-story model structure can be found in (Shirazi et al., 2011), by considering an output feedback controller for MR damper. In (Rotondo et al., 2013), a method to design a quasi-linear parameter varying (quasi-LPV) modeling, identification and control of a Twin Rotor MIMO by considering a state feedback gain-scheduling controller was proposed. A survey about LPV methods for vehicle dynamic control can be consulted in (Sename et al., 2013b). (Bolea et al., 2014) proposed a method to design LPV controllers applied to real-time control of open-flow irrigation canals. Military application to a modern air defense missile model can be consulted in (Tekin and Pfifer, 2013).

Furthermore, LPV models have proved their applicability to design ***Fault diagnosis (FD)*** systems. FD systems are designed to increase not only the performance, but also the safety requirements. These topics have been intensely studied and many methods have been proposed in the last years. The literature contains many FD approaches, e.g. in (Ganguli et al., 2002) the authors present a reconfigurable LPV fault tolerant control for a Boeing 747; the controller was scheduled with a fault signal generated by an FD algorithm in order to guarantee stability and robustness for the closed-loop operation. A robust fault detection system applied to a turbine engine was studied in (Gou et al., 2008); the proposed method was designed by considering thermodynamics variety rate to describe aero engine's dynamic process. A strategy to design a residual generator filter for affine LPV systems was presented in (Armeni et al., 2009). Integrated fault detection and control problem with bounded disturbances was studied in (Wang and Yang, 2009); the method considered robustness against exogenous disturbances and sensitivity against faults based on the residual generation method proposed in (Zolghadri et al., 2008), in order to apply the method to an experimental data set of a secondary circuit of a nuclear power plant. Another feedback controller implemented to the automatically balancing of a riderless bicycle in the upright position was reported in (Andreo et al., 2009). The problem of unknown input reconstruction was well-considered in (Shinar, 2009), the authors demonstrated that when perfect disturbance decoupling is not possible, robustness aspects need to be considered in order to guarantee successful estimation. A virtual sensor as a fault detection and accommodation system based on unknown input observers was studied by (Chadli, 2010). In (Blesa et al., 2011), an FD scheme based on parity equations was formulated as an optimization problem, given by a zonotope space to define adaptive thresholds. An actuator fault tolerant controller was proposed in (Ichalal et al., 2012) by extending the fast adaptive observer for LTI systems studied in (Zhang and Jiang, 2008); in addition, some relaxed stability conditions were obtained by considering Polya's theorem. An FD technique for a fixed-wing aircraft systems was proposed in (Rosa and Silvestre, 2013); the method is based on set-value observers designed in a way that the model uncertainty and disturbances can be considered to avoid false alarms. An FD method, with a quality factor, applied to winding machine under multiple sensor faults was recently proposed in (Rodrigues et al., 2013). More deep reviews can be consulted in the surveys (Alcorta-García and Frank, 1997; Frank et al., 2000; Hwang et al., 2010; Samy et al., 2011) and books (Chen and Patton, 1999; Ding, 2008; Noura et al., 2009).

The previous review illustrates that control techniques for regular LPV systems have become very popular in many domains (H_∞ control, observer design, fault diagnosis, MPC, etc.) and illustrates also the applicability of LPV based methods. However, very few contributions can be found for D-LPV systems as analyzed below.

As detailed before, descriptor systems arise naturally as dynamic models of many applications. Many approaches have been explored for descriptor LTI systems, e.g. state observer design (Darouach and Boutayeb, 1995; Darouach et al., 1996; Hou and Muller, 1999; Madady, 2011), observer-based controllers (Darouach and Zerrougui, 2010), Kalman filter (Sun et al., 2008), fault detection and fault diagnosis (Chen and Patton, 1999; Hwan-Seong et al., 2001), and references therein. Nevertheless, only few works are dedicated to ***descriptor LPV systems***. For example, (Takaba, 2003) proposed some criteria for local stability and quadratic performance applied to feedback systems with component saturation. In (Chadli et al., 2008), the design of static output controllers for singular LPV systems was studied by considering poly-quadratic Lyapunov equations in order to design a static output controller, which was obtained by interpolation of multiple linear static output gains. As a result of the proposed method, a transformation technique of the system was obtained with the objective of reducing the number of LMI constraints and obtaining higher degree of freedom. In (Li et al., 2010), stability conditions for D-LPV systems with time-varying delays were obtained by means of parameter dependent Lyapunov functions. A robust stabilization method based on H_∞ was proposed in (Bouarar et al., 2010), by considering classical closed-loop dynamics and redundancy closed-loop dynamic approaches. It is important to note that the research about D-LPV systems has started in recent years, reason for which some control problems have not been studied extensively. Moreover, the last published works have proved its importance and applicability to real implementations.

Some real ***applications of D-LPV systems*** have been published, e.g. the authors in (Masubuchi et al., 2004) proposed a gain scheduling controller applied to a flight vehicle; the proposed control strategy proved to satisfy several frequency-domain and time-domain specifications. A sensitivity analysis was done using Bode and Nyquist plots to illustrate that the gain-schedulled controller attained all the specifications and the control performance. This work was improved in (Masubuchi and Suzuki, 2008) by removing restrictions of the descriptor realization forced in (Masubuchi et al., 2004). Consequently, sufficient conditions, given by sets of Linear Matrix Equalities (LME) and LMI, were given to deduce a state-space gain-scheduled controller. In (Bouali et al., 2006, 2008), the authors studied a H_∞ state feedback gain scheduling controller showed that it is possible to avoid infinite LMIs, when affine dependence of the parameter appears. In (Guelton et al., 2008), a nonlinear model of human standing was proposed based on a double inverted pendulum; its main contribution was to estimate both the joint torques and the velocities from only the angular position measurements.

Among all the ***fault diagnosis method*** reported in the literature, parity checks (Gertler, 1997), identification methods (Isermann, 1984), fault detection filters (Wang et al., 2007), etc., the *observer-based methods* have become one of the most successful techniques for D-LPV systems. In general, four approaches are commonly considered to design fault detection observers for D-LPV systems. The first approach is based on quadratic Lyapunov functions to guarantee stability of the error equation. Some observers based on this approach can be consulted in (Hamdi et al., 2012a; Rodrigues et al., 2012; Aguilera-González et al., 2013, 2014). The second approach is dedicated to robust observer design by considering H_∞ performance and Lyapunov stability (Marx et al., 2007a; Hamdi et al., 2012b; Shi and Patton, 2014). The third approach considers parameter-dependent Lyapunov functions to verify stability as proposed in (Daafouz and Bernussou, 2001) and applied to descriptor systems in (Astorga-Zaragoza et al., 2011), which gives sufficient conditions of global stabilization and leads to less conservative results. The last approach is related to design fault estimation observers (FEO) as in (Gao et al., 2008; Ghorbel et al., 2012), where FEO were proposed to estimate simultaneously the system state and sensor faults. In the same spirit, a robust FEO was studied in (Bouattour et al., 2011) by considering the H_-/H_∞ technique. Nevertheless, the last approach is in fact dedicated to regular LPV systems and considers the "descriptor approach" by transforming the original into a D-LPV system to estimate sensors faults. Moreover, all the previous approaches consider that the scheduling parameter is measurable, which is not always true and reduces its applicability. In many applications, the scheduling functions depend on the states, e.g. systems modeled by considering the nonlinear sector approach. In such case, the states could be unmeasurable. Unfortunately, the developed methods for systems with measurable scheduling functions are not directly applicable for systems with unmeasurable scheduling functions, and usually a suitable design involves an observer in order to estimate the unmeasurable vector. Note that the problem of unmeasurable scheduling parameter is different than the approach which considers the parameter as inexact or affected by some bounded uncertainties as in (Heemels et al., 2010; Sato and Peaucelle, 2013). The following Section reviews some published works dedicated to the analysis of LPV/D-LPV systems with unmeasurable gain scheduling functions.

2.2.2 Unmeasurable gain scheduling function case

The idea of considering unmeasurable scheduling functions was studied first by (Zak, 1999) to stabilize a multiple model system by considering linear controllers. Following the same idea, a robust controller was studied in (Shi and Nguang, 2003). (Bergsten and Palm, 2000; Bergsten et al., 2002) proposed a method to analyze and design sliding mode observers, which proved to deal effectively with model/plan mismatches. The system was transformed such that the mismatches represented the difference between the unmeasurable and the estimate scheduling function. Indeed, the uncertain system transformation proved to deal successfully with the unmeasurable scheduling problem, e.g. the observer design applied to state estimation was treated in (Yoneyama, 2009). The design of robust observers for fault diagnosis purposes was studied in (Ichalal et al., 2008, 2010), (Theilliol and Aberkane, 2011). Nevertheless, only few contributions have been proposed for D-LPV systems, although , we can cite (Nagy-Kiss et al., 2011), where the authors proposed a state observer by transforming the D-LPV system with unmeasurable scheduling functions into an equivalent uncertain system. In (Nagy Kiss et al., 2011), an unknown input observer was developed by considering the original system as a perturbed system, where the perturbation vector represents a bounded uncertainty given by the measurable and the unmeasurable scheduling functions. In both previous works, the observers were successfully evaluated by using a nonlinear model of a wastewater treatment plant. In the same context, based on the perturbed system technique, a fault detection scheme was proposed in (Hamdi et al., 2012a) with application to an electrical system.

As it can be concluded from the previous review, the problem of control techniques, specially observer design and its application to fault diagnosis, for D-LPV systems with unmeasurable scheduling functions has not been fully investigated so far. Moreover, all these works exemplify the relevance of the techniques for D-LPV systems with unmeasurable scheduling functions and their application to fault detection and isolation. Furthermore, the results proposed in this thesis address these topics by considering four axes:

first, the observer design is addressed by considering the H_∞ performance in order to attenuate disturbances, noise and minimize the error generated by the unmeasurable scheduling functions. Second, fault detection and isolation is addressed by means of residual evaluation. Third, fault diagnosis is performed by extending one of the proposed H_∞ methods. Finally, by combining H_∞ and H_- approaches, a fault detection observer, sensitive to faults and insensitive to disturbances, is proposed.

The following Section presents some methods to model descriptor nonlinear systems as D-LPV systems, to illustrate how this kind of systems, with scheduling functions depending on the states, are obtained, for example considering the nonlinear sector approach.

2.3 Modeling of D-LPV systems

There are different methods to derive a D-LPV system from a nonlinear system, such as linearization (Johansen et al., 2000; Marcos and Balas, 2004), sector nonlinearity approach (Ohtake et al., 2003), state-space transformation (Shamma and Cloutier, 1993), function substitution (Wey, 1997; Pfifer, 2012) and identification (Tóth, 2010). Detailed information about derivation of LPV systems can be found in (Marcos, 2001). Nevertheless, the nonlinear sector approach and the linearization are usually considered, such methods are detailed below.

2.3.1 Linearization

This method is used to obtain LPV/D-LPV from a set of D-LTI plants as proposed in (Johansen et al., 2000). The D-LTI models, which form the D-LPV system, are obtained by linearization around different representative linearization points. Fig. 2.1 illustrate the linearization approach. For a given response of a nonlinear system $y = f(x)$, it is possible to identify a number of linearization points $\mathcal{M}_i$ then, by linearizing the system in each one of these points, a global nonlinear approximation can be obtained. Nevertheless, it must be noted that the linearization points must be judiciously chosen, and different approximations can be obtained depending on the number and selection of these points. The global approximation is given by the interpolation of the local sub-models as

$$y \approx \sum_{i}^{h} \rho_i(\zeta(t)) \left[f(x_i) + f'(x_i)(x - x_i) \right], \; \forall h = 1,2,3. \tag{2.7}$$

where ρ_i are the scheduling functions, as defined in (2.6). This linearization is in fact a Taylor series expansion (truncated) in different representative points, which may or may not be equilibrium. The resulting interpolated global model is a local approximation of the nonlinear plant.

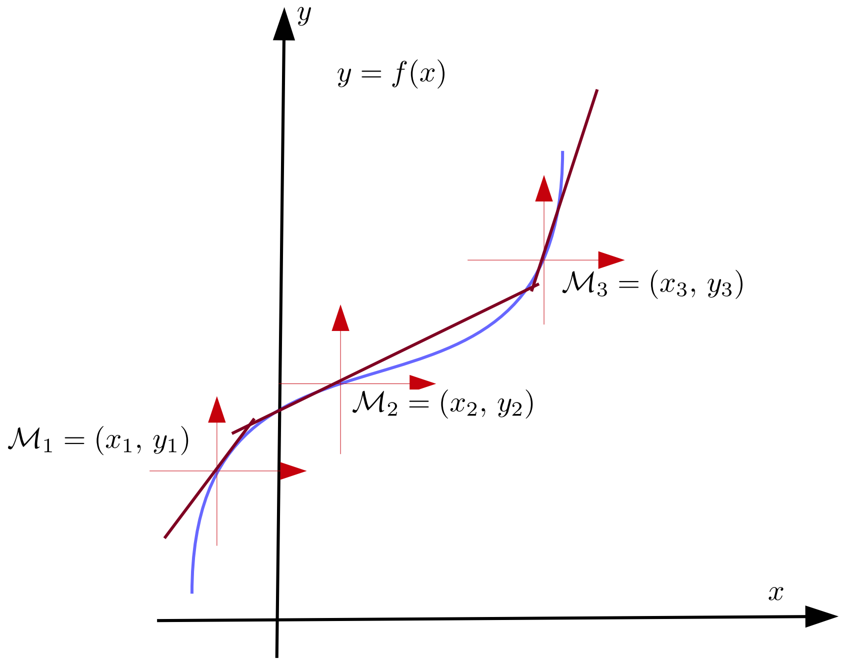


Figure 2.1 – Local approximation

In a more general framework, the aim is to approximate a descriptor nonlinear system (2.3) into a family of D-LTI models described by the following D-LPV approximation

$$E\dot{x}(t) = A_i x(t) + B_i u(t) + \Delta_{xi} \tag{2.8}$$

$$y(t) = Cx(t), \tag{2.9}$$

blended through gain scheduling functions as defined in (2.6) such that the global system is described as

$$E\dot{x}(t) = \sum_{i=1}^{h} \rho_i(\zeta(t)) \{A_i x(t) + B_i u(t) + \Delta_{xi}\} \tag{2.10}$$

$$y(t) = Cx(t), \tag{2.11}$$

where A_i, B_i, and C are state-space matrices of the local models, which are obtained by evaluating the Jacobian matrices at the *ith* operation point as

$$A_i = \left.\frac{\partial f(x,u)}{dx}\right|_{\substack{x = x_i^e \\ u = u_i^e}}, B_i = \left.\frac{\partial f(x,u)}{du}\right|_{\substack{x = x_i^e \\ u = u_i^e}} \tag{2.12}$$

Since generally the linearization is not done in equilibrium points, affine terms must also be added as

$$\Delta_{xi} = f(x_i^e, u_i^e) - A_i x_i^e - B_i u_i^e \tag{2.13}$$

where x_i^e are the values of x at the ith operating point. In the presence of a perfectly linear system, the expression $\Delta_{xi} = 0$.

The scheduling functions are selected such that for any allowed combination of the scheduling variables at least one scheduling function has a true value bigger than zero. Typically, triangular or Gaussian gain scheduling functions are defined to guarantee a smooth interpolation of the local models depending in which operation region is the scheduling vector. Moreover, the scheduling vectors are chosen as the states, which describe the nonlinearities in the Jacobian matrices.

Example 2.3.1. In order to illustrate the method, let us to consider a rolling disc, which rolls on a surface without slipping and is connected to a fixed wall with a nonlinear spring and a linear damper. The mathematical model, in descriptor nonlinear form, is described by the following set of equations (Sjöberg and Glad, 2006):

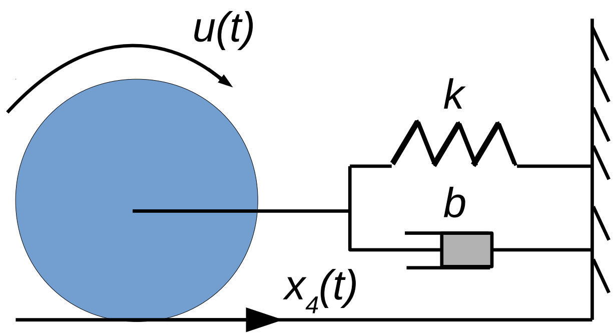


Figure 2.2 – The rolling disc

$$\dot{x}_1(t) = x_2(t) \tag{2.14}$$

$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{k}{m}x_1^3(t) - \frac{b}{m}x_2(t) - \frac{1}{m}x_4(t) \tag{2.15}$$

$$0 = x_2(t) - rx_3(t) \tag{2.16}$$

$$0 = -\frac{k}{m}x_1(t) - \frac{k}{m}x_1^3(t) - \frac{b}{m}x_2(t) + \left(\frac{r^2}{J} + \frac{1}{m}\right)x_4(t) + \frac{-r}{J}u(t) \tag{2.17}$$

were $x_1(t)$ is the position of the center of the disc, $x_2(t)$ is the translational velocity of the same point, $x_3(t)$ is the angular velocity of the disc, $x_4(t)$ is the contact force between the disc and the surface, $u(t)$ is the applied input force to the disk, and $y(t)$ the position.

Table 2.1 – Parameters of the rolling disc system

Parameter	Symbol	Value
Spring coefficients	k	100 N/m
Damper coefficient	b	30
Radius of the disc	r	0.4
Inertia coefficient	J	3.2 kg m^2
Mass	m	40 kg

From (2.2), it is possible to identify the nonlinear term, which is $x_1(t)$. Therefore, $x_1(t)$ is chosen as the scheduling vector $\zeta(t) = x_1(t)$. Two different equilibrium points are selected in $x_1^{e_1} = -0.2$ m and $x_1^{e_2} = 0.27$ m/s, [1]. Consequently, triangular gain scheduling functions are designed as

$$\rho_1 = 1 - \tanh x_1(t);\, \rho_2 = 1 - \rho_1$$

such that for any t the two models are activated as displayed in Fig. 2.3. The state-space matrices are obtained, first by deriving the Jacobian matrices of (2.14) as

$$A_i = \frac{\partial f(x,u)}{dx}\bigg|_{\substack{x = x_i^e \\ u = u_i^e}} = \left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{m}\left(1+3x_1^2(t)\right) & \frac{-b}{m} & 0 & \frac{1}{m} \\ 0 & 1 & -r & \\ \frac{-k}{m}\left(1+3x_1^2(t)\right) & \frac{-b}{m} & 0 & \left(\frac{r^2}{J}+\frac{1}{m}\right) \end{bmatrix}\right|_{\substack{x = (x_1^{e_i},0,0,0) \\ u = 0}}$$

$$B_i = \frac{\partial f(x,u)}{du}\bigg|_{\substack{x = x_i^e \\ u = u_i^e}} = \left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-r}{J} \end{bmatrix}\right|_{\substack{x = (x_1^{e_i},0,0,0) \\ u = 0}}$$

$$\Delta_{xi} = f(x_i^e, u_i^e) - A_i x_i^e - B_i u_i^e = \left.\begin{bmatrix} 0 \\ -\frac{2k}{m}x_1^3(t) \\ 0 \\ -\frac{2k}{m}x_1^3(t) \end{bmatrix}\right|_{\substack{x = (x_1^{e_i},0,0,0) \\ u = 0}}$$

[1] Note that the selection of the operation point is quite heuristically, and there is no method to define how many points are necessary to describe efficiently the nonlinear system. However, previous knowledge about the evolution of the nonlinear state, for example the bounds, can be considered to select the operation points by considering points contained within the bounds.

Secondly, the state-space matrices are computed evaluating the Jacobian matrices at the three operation points, such

$$E = \mathsf{diag}(1, 1, 0, 0)$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2.8 & -0.75 & 0 & 0.025 \\ 0 & 1 & -0.4 & 0 \\ -2.8 & -0.75 & 0 & 0.075 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2.5 & -0.75 & 0 & 0.025 \\ 0 & 1 & -0.4 & 0 \\ -2.5 & -0.75 & 0 & 0.075 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.1250 \end{bmatrix}, \Delta_{x1} = \begin{bmatrix} 0 \\ -0.04 \\ 0 \\ -0.04 \end{bmatrix}, \Delta_{x2} = \begin{bmatrix} 0 \\ 0.0984 \\ 0 \\ 0.0984 \end{bmatrix}$$

A simulation is done in order to show the approximation quality. Initial conditions are choose as $x(0) = [0.1, 0.3, 0.75, 9.7]$, the input signal is defined as

$$u(t) = \begin{cases} t-2 & t \leq 2 \\ 0 & otherwise \end{cases} \tag{2.18}$$

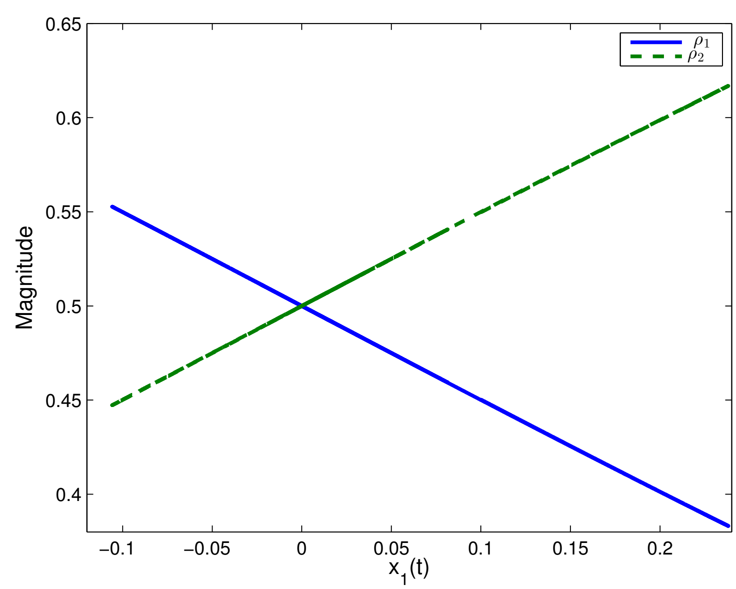


Figure 2.3 – Gain scheduling functions

As defined previously, the scheduling functions depend on the state $x_1(t)$. The evolution of the scheduling

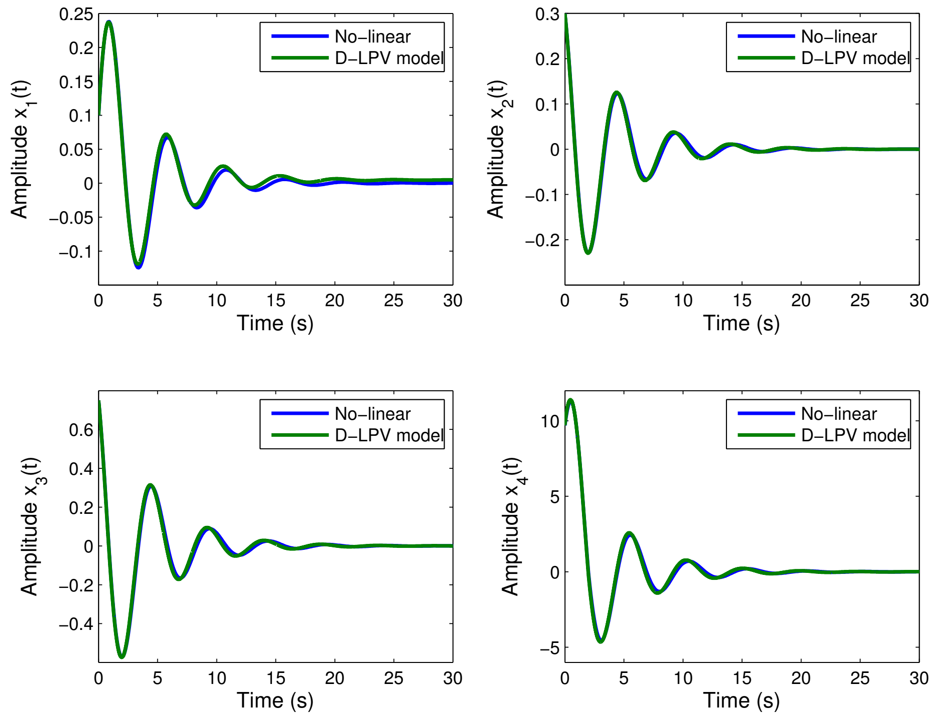


Figure 2.4 – Dynamic behavior of the states for the nonlinear system and the D-LPV model by considering the linearization approach

functions are as displayed in Fig. 2.3. The evolution of the nonlinear and D-LPV system states are show in Fig. 2.4. As it can be deduced from the figures, the linearization approximates the nonlinear system, despite the small error presented between the states.

The example shows that the D-LPV model approximates the nonlinear system. However, its main drawback is that there is no general method to select the linearization points (and how many). A more accurate D-LPV representation of the nonlinear system can be obtained by considering the sector nonlinearity approach detailed in the next section.

2.3.2 The sector nonlinearity approach

The nonlinear sector approach, also known as polytopic transformation method, was proposed by (Ohtake et al., 2001). The main idea is to obtain a D-LPV systems, in its quasi-D-LPV form, such that the global model matches exactly the nonlinear system in a compact set of the state variable. The number of sub-models is directly related to the number of the nonlinear terms. For each nonlinear term, two sub-models are obtained such that for k nonlinear terms, the global model is composed of $h = 2^k$ sub-models.

Therefore, while bigger the number of nonlinear terms in the nonlinear-descriptor system, the bigger is the conservatism in the global D-LPV system.

To illustrate the method, consider Fig. 2.5a and Fig. 2.5b that describe two different nonlinear systems of one variable given by $\dot{y} = f(x)$ where $f(0) = 0$. In some cases it is possible to enclose all the dynamic of a nonlinear system by finding a global sector such that $\dot{y} = f(x(t)) \in [s_1 \; s_2]$, where s_1 and s_2 are lines that enclose the sector, as shown in Fig. 2.5a. A global sector guarantees an exact representation of the nonlinear model. However, it is not always possible to find a global sector. In that case, it is possible to find a local sector bounded by the region $\underline{a} < x(t) < \bar{a}$, as shown in Fig. 2.5b, and then represent the nonlinear system in the local sector $\underline{a} < x(t) < \bar{a}$.

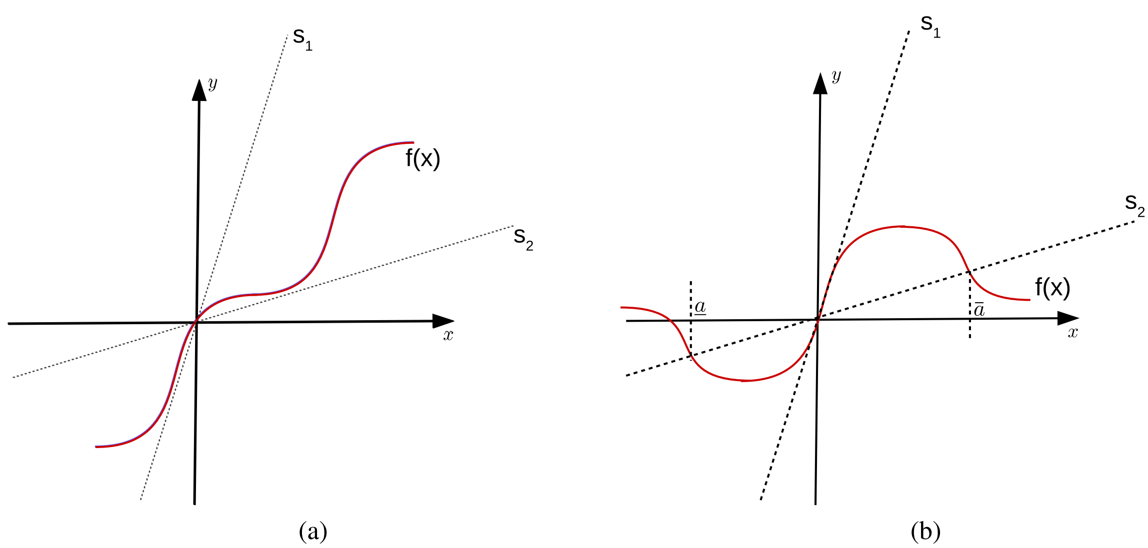


Figure 2.5 – The sector non-linearity approach: a) Global sector ; b) Local sector

The D-LPV representation is obtained by hiding the nonlinearities in the scheduling variables such that for any nonlinear term which is represented by $\zeta_k \in [\bar{a}_k, \underline{a}_k], \forall k = 1, 2..l$, two gain scheduling functions are designed as

$$\mu_1^k(\zeta(t)) = \frac{\bar{a}_k - \zeta_k}{\bar{\zeta}_k - \underline{a}_k}, \mu_2^k = 1 - \mu_1^k \tag{2.19}$$

These two weighting functions are normalized such that $\mu_1^k > 0$, $\mu_2^k > 0$, and $\mu_1^k + \mu_2^k = 1$ for any value of ζ_k. The membership function of rule ith is computed as the product of the weighting functions that correspond to the local model

$$\rho_i(\zeta(t)) = \prod_{k=1}^{l} \mu_i^k(\zeta_i) \tag{2.20}$$

where μ_{ik} is either μ_1^k or μ_2^k, depending on which local weighting function is considered. The scheduling functions are normal $\rho_i(\zeta(t)) \geq 0, \sum_{i=1}^{h} \rho_i(\zeta(t)) = 1$. Finally, a system as given in (2.5) is obtained by evaluating the state-space matrices in the chosen sectors $\underline{a} < x(t) < \bar{a}$. Note that this *quasi-LPV* representation constitutes a polytopic form because the matrices with variable parameters are convex combinations of constant matrices calculated from the polytope vertex (Nagy-Kiss et al., 2011).

The following example illustrates the steps to obtain a D-LPV system.

Example 2.3.2. Consider the rolling disc system given in Example 2.3.1 rewritten as follows

$$E\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m}\left(1+x_1^2(t)\right) & \frac{-b}{m} & 0 & -\frac{1}{m} \\ 0 & 1 & -r & 0 \\ -\frac{k}{m}\left(1+x_1^2(t)\right) & \frac{-b}{m} & 0 & \left(\frac{r^2}{J}+\frac{1}{m}\right) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-r}{J} \end{bmatrix} u(t) \tag{2.21}$$

The scheduling variable is chosen as the non-constant term in (2.21) such that $\zeta(t) = x_1^2(t)$ which is bounded in $-0.15 < x_1(t) < 0.28$. The corresponding scheduling functions are computed as follows

$$\mu_1(z(t)) = \frac{1-\zeta_i}{\zeta_i+1}, \mu_2 = 1-\mu_1 \tag{2.22}$$

The linear models are obtained by substituting the bounded values of ζ in (2.21), such that

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2.1250 & -0.75 & 0 & 0.025 \\ 0 & 1 & -0.4 & 0 \\ -2.1250 & -0.75 & 0 & 0.075 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.2 & -0.75 & 0 & 0.025 \\ 0 & 1 & -0.4 & 0 \\ -3.2 & -0.75 & 0 & 0.075 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.1250 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, E = \mathrm{diag}(1, 1, 0, 0), \end{aligned}$$

where C is the output matrix chosen as convenience.

In order to validate the model, a simulation is done by considering the same initial conditions, as given in the Example 2.3.1. As displayed in Fig. 2.6, the D-LPV model matches well the nonlinear systems. The scheduling functions, as displayed in Fig. 2.7, shown the soft interpolation between the two model in order to reproduce the nonlinear behavior.

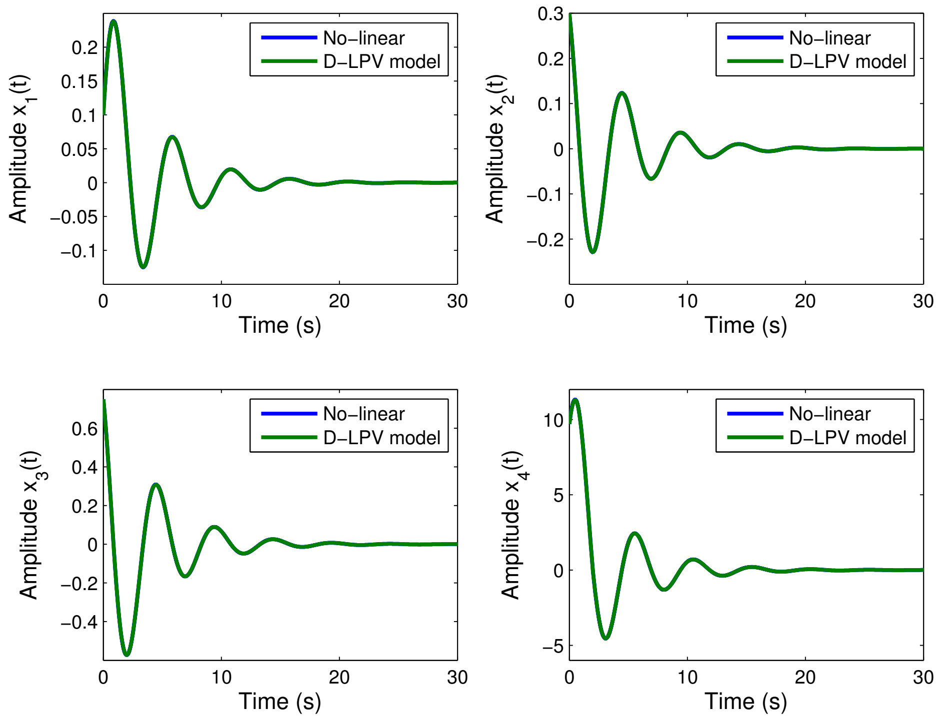


Figure 2.6 – States comparison between the nonlinear system and the D-LPV model by considering the sector nonlinearity approach

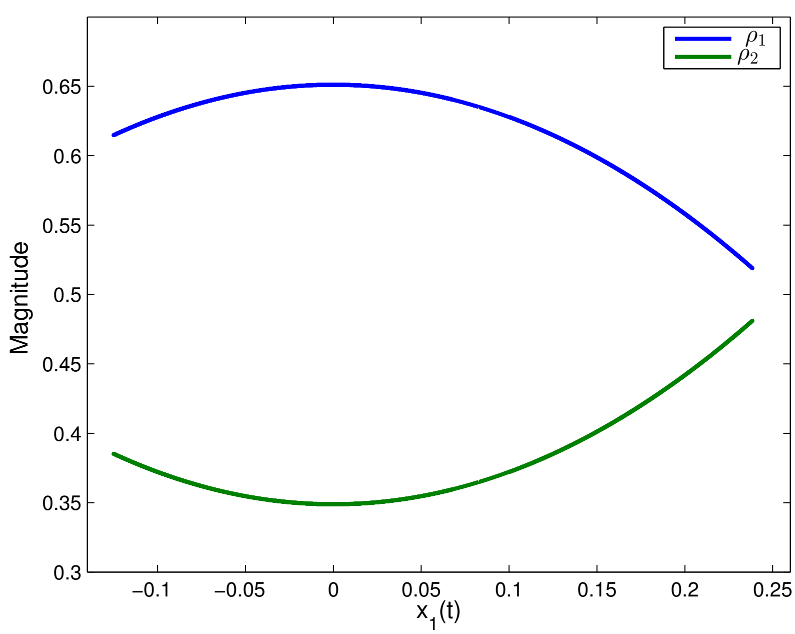


Figure 2.7 – Gain scheduling functions

Table 2.2 – State RMS errors between the nonlinear and the D-LPV systems

	error x_1	error x_2	error x_3	error x_4
Linearization	0.0002	0.0001	0.0004	0.0455
Sector nonlinearity	0.0001×10^{-4}	0.0001×10^{-4}	0.0015×10^{-4}	0.2418×10^{-4}

The main advantage of the sector nonlinearity approach is that the D-LPV model is an exact representation of the nonlinear systems in the selected sector. This can be seen clearly in Table 2.2 where the states RMS errors between the nonlinear systems and the D-LPV models are displayed. Note that despite both models match the nonlinear system, the sector nonlinearity approach fits the nonlinear model with smallest errors. This is why the sector nonlinearity has become a more popular method over the linearization method. However, despite its systematic procedure to obtain the D-LPV representation, the consequent model can be unstable or unobservable, which can reduce its applicability to design control or supervision systems. This problem can be solved by considering a detailed selection of the scheduling variables and a careful selection of the upper and lower bounds of the nonlinear sector that are not trivial, and they should be determined by a combination of previous expertise and trial and error procedure. In this respect, the proposed approach can be viewed as an effective procedure that requires some knowledge of the system to be modeled. Therefore, for the applicability of the mathematical model, it is always necessary to verify its stability, controllably or observability in order to design a suitable control system. In particular, the proposed methods in this thesis require that the models are observable and stable. These properties are studied in detail in the next Section.

2.4 Properties of D-LPV systems

This Section is dedicated to the study of some assessing properties of D-LPV systems, such as stability and observability, which are important to design state observers. Other properties, as impulse controllability and solvability, which are necessary for control purposes, are not analyzed in this section, but can be consulted in Appendix A. Moreover, this Section details first some properties and definitions for D-LTI systems and then, extends them to D-LPV system. Therefore, we encourage the readers to read both subsections in order to have a full understanding of the properties definitions.

2.4.1 Stability and admissibility of D-LTI systems

In standard state-space systems, for any initial condition $x(0)$ and for any known input $u(t) \in [0,t]$, the response is unique. For descriptor systems, the existence and uniqueness of a solution is valid only if the system is regular. The following definitions express regularity properties for the continuous D-LTI system (2.4), rewritten here by convenience:

$$\begin{cases} E\dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{cases} \tag{2.23}$$

Definition 2.4.1. (Gantmacher, 1959) The system (2.23) is called regular if there exists a constant scalar γ such that $\det(\gamma E - A) \neq 0$ or, equivalently, the polynomial $\det(sE - A)$ is not identically zero. In this case the matrix pair (E, A), or the matrix pencil $sE - A$; is regular.

Although the regularity condition seems restrictive, this is not a very restrictive assumption and most occurring practical systems are regular (Sjöberg and Glad, 2006). Then, for a regular descriptor system, the following stability condition holds:

Theorem 2.4.1. (Dai, 1989) A regular descriptor system is stable if and only if

$$\mathrm{eig}(E, A) \subseteq \mathbb{C}^- := \mathrm{eig}(sE - A) = \{s | s \in \mathbb{C}, \mathrm{Re}(s) < 0\} \tag{2.24}$$

From the theorem, it is deducible that stability is defined by the location of the eigenvalues in the complex plane. If the eigenvalues have negative real part, then the system is stable. If the system eigenvalues have zero real part, then it is critically stable, and if the eigenvalues have positive real part, then the system is unstable.

Another property associated to the eigenvalues of the pair (E, A) are the impulse modes. Standard state-space systems, with $E = I$, only have finite modes. However, a descriptor system may have both finite and infinite modes. The infinite modes are related to infinite eigenvalues of $(sE - A)$ or equivalent eigenvalues $\lambda = 0$ to $(E - \lambda A)$. Clearly, the infinite eigenvalues are related to the non-dynamic part corresponding to the algebraic equations. Therefore, the number of non-dynamic modes of (2.23) is given by the total number of states minus the rank of E, i.e. $n - r$, where r is the number of dynamic modes. The dynamical modes incorporate both, the finite and the infinite modes. The number of finite modes is computed by

$$n_\iota = \deg\det(sE - A) \tag{2.25}$$

In other words, the number of infinite modes is expressed by $r - n_\iota$. In the case of $r = n_\iota$, then it is equivalent to say that the system contains zero infinite modes, or equivalently impulse modes equal to zero, in which case the system is impulse free or index one. The index of (2.23) can be defined as:

Definition 2.4.2. The number of differentiations needed in order to get a nonsingular matrix E is called the index of (2.23).

Special methodology to compute the index, as the Shuffle algorithm, can be consulted in (Luenberger, 1978; Pantelides, 1988; Dai, 1989; Campbell, 1995). Nevertheless, among different methods, the easiest way to compute the index of a DAE system is by the Kronecker-Weierstrass restricted system equivalent transformation (r.s.e), also known as the canonical form of (2.23), which is defined as follows:

Definition 2.4.3. (Dai, 1989) Two systems (E, A, B, C) and $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ are defined as equivalent, or restricted systems equivalent (r.s.e.), if their order, number of inputs and outputs are equal and there exist two non singular matrices P and Q such that $\hat{E} = PEQ$, $\hat{A} = PAQ$, $\hat{B} = PB$, $\hat{B} = CQ$, where

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{bmatrix} \tag{2.26}$$

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{r_\iota} \end{bmatrix} \tag{2.27}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$, I_{n_1} is the identity matrix, and $\mathcal{N}$ is a nilpotent matrix of index r_ι such that $\mathcal{N}^{r_\iota} = 0$ but $\mathcal{N}^{r_\iota - 1} \neq 0$. The equivalent system, also called the restricted system equivalent (r.s.e.), is:

$$\left.\begin{aligned} \dot{x_1} &= A_1 x_1(t) + B_1 u(t) \\ y_1(t) &= C_1 x_1(t) \end{aligned}\right\} \text{Slow subsystem} \tag{2.28}$$

$$\left.\begin{aligned} \mathcal{N}\dot{x_2} &= x_2(t) + B_2 u(t) \\ y_2(t) &= C_2 x_2(t) \end{aligned}\right\} \text{Fast subsystem} \tag{2.29}$$

or equivalently

$$\begin{bmatrix} I & 0 \\ 0 & \mathcal{N} \end{bmatrix} \begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{2.30}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB, \quad [C_1 C_2] = CQ$$

Subsystem (2.28) represents the *slow response, the dynamic part,* and subsystem (2.29) represents the *fast response, the algebraic part. Other r.s.e transformations can be consulted in Appendix* (A.5).

It is clear from previous definition that if the degree of nilpotency $\mathcal{N} = 1$, then the system is impulse free. As we comment previously, in practice almost all the systems are impulse free (equivalent index=1). However, when the index is greater than one, it is possible in many situations to transform a system of index r_ι to a system of index one. Some examples of index transformation can be consulted in (Pantelides, 1988; Sjöberg and Glad, 2006).

By considering previous definition, it is possible to generalize impulse free and stability conditions, which rely on the following admissibility condition

Definition 2.4.4. (Dai, 1989) The system (2.23), or the pair pencil (E,A), is called admissible if it is stable and impulse-free.

Remark 2.4.5. Previous definitions are given for D-LTI systems. These conditions can be considered to verify regularity, and stability for D-LPV system (2.5) in each vertex (Masubuchi and Suzuki, 2008; Chadli and Darouach, 2011), that can be sufficient from a practical point of view. Nevertheless, a more generalized stability condition can be obtained in terms of LMIs, which involve no decomposition of the system matrices and guaranteed stability for all convex combinations, rather than the vertex. This method is detailed in the next Section.

2.4.2 Stability of D-LPV systems

Consider the autonomous D-LPV system ($u(t) = 0$) as

$$\begin{cases} E\dot{x}(t) & = \sum_{i=1}^{h} \rho_i(\zeta(t)) \{A_i x(t)\} \\ y(t) & = Cx(t) \end{cases} \tag{2.31}$$

The following lemma gives conditions for stability

Lemma 2.4.1. *The system (2.31) is said to be stable with (E,A_i) admissible, if there exists a matrix P such that $\forall i \in [1,2 \dots h]$, the following hold*

$$E^T P = P^T E \geq 0 \tag{2.32}$$

$$A_i^T P + P^T A_i \leq 0 \tag{2.33}$$

Proof. Consider a Lyapunov function $V(x(t)) = x^T(t) E^T P x(t)$, with $E^T P = P^T E \geq 0$, and whose derivative is

$$\dot{V}(x(t)) = \dot{x}^T(t) E^T P x(t) + x^T(t) E^T P \dot{x}(t) < 0 \tag{2.34}$$

By considering condition (2.32), the following is equivalent

$$\dot{V}(x(t)) = (E\dot{x}(t))^T Px(t) + x^T(t)P^T E\dot{x}(t) < 0, \tag{2.35}$$

such that

$$\dot{V}(x(t)) = \left(\sum_{i=1}^{h} \rho_i(\zeta(t)) \{A_i x(t)\}\right)^T Px(t) + x^T(t)P^T \left(\sum_{i=1}^{h} \rho_i(\zeta(t)) \{A_i x(t)\}\right) < 0$$
$$\dot{V}(x(t)) = \sum_{i=1}^{h} \rho_i(\zeta(t)) x^T(t) \left(A_i^T P + P^T A_i\right) x(t) < 0 \tag{2.36}$$

Therefore, the feasibility of (2.36) gives sufficient conditions for stability, as defined in Lemma 2.4.1. □

Lemma 2.4.1 gives sufficient conditions to guarantee stability of system (2.31). Furthermore, other stability criteria can be consulted in (Chadli et al., 2014).

2.4.3 Observability

Different from regular LPV system, D-LPV systems have different types of observability related to the differential and the algebraic equations. In any case, the observability of a D-LPV system is concerned with the ability to reconstruct the state from the system inputs and the measured outputs. The ability to reconstruct only the reachable state from the output data is characterized by the Reachable-observability (R-Observability). On the other hand, due to algebraic equations, impulsive terms can appear. Impulsive terms are not desirable since they can saturate the state response or, in general, they can have negative effects on the system. Impulse-observability (I-observability) guarantees the ability to estimate impulse terms given by the algebraic equations. So, to design a state LPV observer, as will be explained in the next Section, a D-LPV system needs to be R and I observable. However, observers for D-LPV systems are designed such that each rule has a local gain and it is required that the local models are observable or detectable. Therefore, to design observers, it is necessary to assume that the local models, are R/I observable (Kamidi, 2000; Lendek et al., 2011).

The following Assumptions express observability properties of D-LPV systems:

Assumption 1. *([Hamdi et al., 2012b]) The triplets (E, A_i, C) of system (2.5) are called observable on the reachable set (R-observable) if*

$$\mathsf{rank}\begin{bmatrix} sE - A_i \\ C \end{bmatrix} = n, \ \forall s \in \mathbb{C}, s\mathit{finite}, \forall i \in [1, 2 ... h]. \tag{2.37}$$

Assumption 2. *(Hamdi et al., 2012b) The triplets (E, A_i, C) of system (2.5) are called impulse observable (I-observable) if*

$$\mathsf{rank}\begin{bmatrix} E & A_i \\ 0 & E \\ 0 & C \end{bmatrix} = n + \mathsf{rank}\, E, \ \forall s \in \mathbb{C}, s \forall i \in [1, 2, ..., h]. \tag{2.38}$$

Remark 2.4.6. The method presented here is also called "the direct approach" to compute observability. However, the method is not unique and different methods, as based on *Laurent expansion* (Karampetakis and Vologiannidis, 2004), can be considered. Other methods and also other properties as controllability are detailed in Appendix A. These methods firstly proposed for D-LTI systems can be extended to D-LPV systems.

All previous definitions were presented in order to introduce the observer design for D-LPV systems that is the main topic in this research. These conditions, stability and observability, are necessary for the methods developed in this thesis to perform state estimation and its application to fault diagnosis. The next Section presents some remarks related to the design of state observers for D-LPV systems with unmeasurable scheduling functions.

2.5 State observers for D-LPV systems with unmeasurable gain scheduling

Many real world applications have some variables that are not measured. Nevertheless many control techniques are based on the assumption that states are available, that is not always true. Although in practice, it is possible to have some information about the input $u(t)$ and the output $y(t)$ rather that the state. In such case, the observability properties, as detailed in section 2.4.3, establish that when the system is R/I observable, the initial state can be determined and, therefore, the state trajectory can be reconstructed from input and output measurements, which can involve a state observer. State observers are dynamic systems that for an observable state equation can be designed to asymptotically estimate the state vector $x(t)$.

There are many works related to the study of observers for D-LTI systems; Luenberger observer based on restricted system equivalence decomposition were studied in (Hou and Muller, 1995; Darouach and Boutayeb, 1995; Darouach et al., 1996), based on Sylvester equation by considering also impulsive modes (Wang, 2001), proportional-derivative (PD) (Zhiwei, 2005; Wu and R., 2007), proportional integral (Koenig and Mammar, 2002), fault detection observers (Hwan-Seong et al., 2001; Koenig, 2005; Kyeong et al., 2005), H_∞ observers based on LMI framework (Darouach, 2009; Darouach and Zerrougui, 2010; Darouach and Benzaouia, 2010), discrete-time observers (Dai, 1988; Takahashi et al., 1999; Sun et al., 2008; Madady, 2011), among others.

Nevertheless, despite the developments done for D-LTI systems, only few on them have been extended to D-LPV systems. For example, proportional observer (Marx et al., 2007b; Nagy-Kiss et al., 2011), observers based on restricted system equivalence decomposition (Hamdi et al., 2009), proportional integral observers (Hamdi et al., 2012b; Aguilera-González et al., 2013), Lipschitz observers (Boulkroune et al., 2013), by considering the descriptor approach [2] (Chadli et al., 2013a; Bouattour et al., 2011; Tong et al., 2011). A more detailed review can be consulted in Section 2.2.

2.5.1 Definitions

The following definition describes mathematically the concept of state observer for D-LPV systems.

Definition 2.5.1. Consider the following descriptor system with disturbances

$$\begin{cases} E\dot{x}(t) & = \sum_{i=1}^{h} \rho_i(\zeta(t)) \{A_i x(t) + B_i u(t) + B_{di} d(t)\} \\ y(t) & = C_i x(t) + D_d d(t) \end{cases} \tag{2.39}$$

where $d \in \mathbb{R}^q$ is the disturbance vector, B_{di} and D_d are constant matrices of appropriate dimensions. If system (2.39) is stable and R and I observable, then consider also, the dynamical state observer

$$\begin{cases} \hat{E}\dot{z}(t) & = \sum_{i=1}^{h} \rho_i(\zeta(t)) \{N_i z(t) + G_i u(t) + L_i y(t)\} \\ \hat{x}(t) & = z(t) + T_2 y(t) \end{cases} \tag{2.40}$$

where $z(t)$ represent the estimated vector, $\hat{x}(t)$ the estimated states. $\hat{E}, N_i, G_i, L_i$ and U are constant matrices of appropriate dimensions. If $\|e(t)\| = \|x(t) - \hat{x}(t)\|$ tends asymptotically to zero when $t \to \infty$, regardless of the unknown input, the control input and the initial conditions $x(0), z(0)$, then the system (2.40) is called a state observer for the system (2.39).

[2] The so called "Descriptor approach" is a methodology considered to estimate sensor faults by transforming a state-space system into a descriptor system such that the algebraic equations in the resulting augmented states represent the sensor fault (Zhang and Jiang, 2008).

If the $\mathrm{rank}(\hat{E}) < n$, the state observer (2.40) is called a descriptor observer; otherwise, if $\mathrm{rank}(\hat{E}) = n$, then without loss of generality it is assumed that $\hat{E} = I_n$, the state observer (2.40) is called a standard observer and takes the following form

$$\begin{aligned}\dot{z}(t) &= \sum_{i=1}^{h} \rho_i(\zeta(t))\,\{N_i z(t) + G_i u(t) + L_i y(t)\}\\ \hat{x}(t) &= z(t) + T_2 y(t)\end{aligned} \tag{2.41}$$

On the other hand, a descriptor-LPV observers is often written as

$$E\hat{x}(t) = \sum_{i=1}^{h} \rho_i(\zeta(t))\,\{A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - C\hat{x})\}$$

where $\hat{x}(t)$ is the estimated states and L_i are the state observer gain matrices.

Clearly, the standard observer is a regular LPV observer commonly used for regular LPV systems. However, it is common and recommendable to consider a standard observer rather than a descriptor observer because its design is straightforward and is easier to implement in practice. Moreover, as studied in (Dai, 1989), descriptor observers are more sensitive to high-frequency noise. However, standard observers have a disadvantage related to number of gain matrices, which can increase the conservatism of solutions given for example in the LMI formulation.

In this Thesis, the problem of the design of observers for both descriptor and standard state observers for D-LPV systems is considered. These methods will be detailed in Chapter 3 and Chapter 4. In addition, we consider the problem of D-LPV systems with unmeasurable scheduling functions. In order to understand more clearly this problem, the following section provides some concepts to illustrate the unmeasurable scheduling case.

2.5.2 Observer design

The classical approach to design state observers considers that the local models of system (2.39) and the observer (2.41) are interpolated by the same gain scheduling functions $\rho_i(\zeta(t))$. Nevertheless, as detailed in Section 2.2.2, in some cases the scheduling functions are unmeasurable and have to be estimated. In such cases, the observer takes the form given below

$$\begin{cases} \dot{z}(t) & = \sum_{i=1}^{h} \rho_i(\hat{\zeta}(t)) \{N_i z(t) + G_i u(t) + L_i y(t)\} \\ \hat{x}(t) & = z(t) + T_2 y(t) \end{cases} \tag{2.42}$$

where $z(t)$ represent the estimated vector, $\hat{x}(t)$ the estimated states. N_i, G_i, L_i, T_2 are unknown matrices of appropriate dimensions, and $\rho_i(\hat{\zeta}(t))$ are the scheduling functions, which depend on the estimated vector $\hat{\zeta}(t)$. Then, the goal is to determine the gain matrices of (2.42) such that the estimation error $e(t) = x(t) - \hat{x}(t)$ converge asymptotically towards zero, despite the unmeasurable scheduling vector.

As in the measurable scheduling case, in the unmeasurable case, the state-space error equation is obtained by connecting system (2.39) and observer (2.42). As a result, the factorization by the activation functions, in the evaluation of the dynamic or error[3], is expressed by

$$\dot{e}(t) = \overbrace{\sum_{i=1}^{h} \rho_i(\zeta(t))}^{\text{unmeasurable}} T_1 [A_i x(t) + B_i u(t) + B_{di} d(t)] - \overbrace{\sum_{i=1}^{h} \rho_i(\hat{\zeta}(t))}^{\text{estimated}} [N_i z(t) + G_i u(t) + L_i y(t)] \tag{2.43}$$

with

$$[T_1 \; T_2] = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}.$$

However, due to the fact that (2.39) and (2.42) are depending on different scheduling vector $\zeta(t)$ and $\hat{\zeta}(t)$, the state-space error (2.43) should be affected by such difference. In other words, a suitable observer design must involve the estimation of the scheduling functions by first estimating the states and then re-injecting the estimated vector into the scheduling functions. This consideration increases the complexity and requires to develop methodologies, different than the measurable gain scheduling case, in order to guarantee robustness and convergence. As discussed in Section 2.2.2, this problem has been addressed mainly for regular LPV systems (Yoneyama, 2009)(Ichalal et al., 2010), (Theilliol and Aberkane, 2011), rather than D-LPV systems (Nagy-Kiss et al., 2011). This problem will be addressed in this Thesis.

2.6 Descriptor System Package

This Section is dedicated to present a computational tool developed as a result of the study of the state of the art of regular LPV and D-LPV systems. Different scripts were developed due to the lack of special computational tools for the analysis and the design of state observers and fault detection methods. As a

[3]The expression (2.43) is given only for illustrative purpose. A detailed synthesis of the error system will be given in Chapter 3 together with the computation of matrices T_1 and T_2.

result, a toolbox for SCILAB and MATLAB was developed. The SCILAB version can be downloaded from ATOMS web page (López-Estrada et al., 2011) and the MATLAB version from the shared files section in the MATLAB web page. In addition, this toolbox was presented in the *2012 workshop on advanced control and diagnosis* (López-Estrada et al., 2012).

To the best of the author's knowledge, only one toolbox has been proposed in the literature (Varga, 2000). The toolbox was developed for MATLAB, and it is not free distributing, and it is focused mainly on the solution of numerical problems. Unlike (Varga, 2000), the developed toolbox can be considered for i) analysis of fundamental properties, ii) observer design and iii) fault detection for D-LTI, LPV, and D-LPV systems. These functions are summarized in the following Table:

Function	Description
Analysis of properties	
dss2tf	DLTI to transfer function
dcontr	C, R and I controllability matrices
dobsv	C, R and I observability matrices
dstabil	Stability
qrrse	r.s.e based in QR decomposition
invrse	Inverse r.s.e
kwrse	Kronecker-Weistrauss r.s.e† *
dc2dd	Continuous to discrete-time *
fundmatrix	Laurent expansion*
Observers design	
darobsv95	Full order observer†
redobsv95	Reduced order observer †
darobsv96	Reduced order observer with unknown inputs†
puiobsv	Proportional UIO (P-UIO)
piuiobsv	Proportional-integral UIO (PIUIO)
LPVpuiobsv	P-UIO for DLPV systems
LPVpiuiobsv	PI-UIO for DLPV systems
Fault detection for DLTI and DLPV	
gosbank1/2	GOS bank using a P-UIO and PIUIO
dosbank1/2	DOS bank using a P-UIO and PIUIO
lpvgosbank1	GOS bank using a P-UIO
Others	
lpvweig2/3/4	Weighting functions of 2,3 or 4 vertex
	†Only for DLTI

Function	Description
	*Only for SCILAB

Some restrictions are listed below

- As it can be seen in the above Table, the descriptor package can be used like an auxiliary tool in the analysis, observer design for DLTI and DLPV, and observer-based fault detection schemes (Frank, 1990). However, the algorithms are limited only to compute the gains of each one of the observers and the implementation of such banks have to be manual.
- Some functions are only available for SCILAB due to the fact that some requirements are not supported by MATLAB.
- The *YALMIP* toolbox (Lofberg, 2004) is necessary to solve some of the LMIs in the MATLAB package. By default, the solver *lmilab* is selected. However, this can be changed by *sedumi* or other solvers.

2.7 Conclusions

This chapter was dedicated to present a detailed study of the state of the art corresponding to D-LPV systems. Some important properties to design LPV observers have been also analyzed, such as stability and observability. Different methods to derive a D-LPV system from nonlinear system have been detailed in depth. In addition, some related works published in the literature have been reviewed to illustrate which are the differences with the main thesis contributions that will be presented in the sequel of the document. The main contributions of this chapter can be summarized as follows:

- Feasible stability conditions have been proposed given by Corollary (2.4.1).
- Three different D-LPV models have been proposed to illustrate the effectiveness of the modeling methods: the linearizion and the sector nonlinearity approach.
- As a result of the review of the state of the art, a toolbox for Scilab/matlab was developed. The toolbox is continuously maintained since 2012 and has been downloaded, in its Scilab version, more than 2600 times. [4]

[4] Matlab version: http://www.mathworks.fr/matlabcentral/fileexchange/40589-descritor-lti-and-lpv-calculation-tool-kit-v-1-21.
Scilab version: downloaded from the ATOMS platform with more of 2600 downloads.

Moreover, as remarked in this chapter, in particular in Section 2.2 and Section 2.5, the study of robust state observer for D-LPV systems with unmeasurable scheduling functions has not yet been fully tackled. Therefore, this topic remains open and challenging. The following chapters of this thesis are dedicated to propose different methods to design this kind of observers and apply them to fault diagnosis.

Chapter 3

Robust H_∞ state observer design for D-LPV systems with application to fault detection and fault estimation

Contents

This chapter is dedicated to the study of the observer design based on the H_∞ approach for D-LPV systems with unmeasurable gain scheduling functions. A general introduction about H_∞ performance is given in Sec. 3.1. After that, a descriptor state observer is studied in Sec. 3.2 by transforming the system with unmeasurable scheduling function into an uncertain system with estimated scheduling functions. Section

3.3 is dedicated to the design of a standard LPV observer, which considers a different transformation to obtain an uncertain system with estimated gain scheduling functions. In Section 3.4, the design of a robust state observer is proposed by considering a transformation which takes advantages of the convex property related to the gain scheduling functions. A common example to compare and show the applicability and performance of each approach is analyzed in Sec. 3.5. Sec. 3.6 is dedicated to fault detection and isolation based on the proposed state observers. In Sec. 3.7, fault diagnosis is considered. A method to estimate sensor faults is proposed by extending the approach presented in Sec. 3.3. The applicability of the methods are studied and illustrated through simulation examples.

3.1 Preliminary definitions

Dynamical systems can be affected by external disturbances and noise that can lead to the loss of stability and performance. A stable system has to remain without loss of performance, despite the disturbances. This property is called robustness. In other words, robustness means that the systems must remain stable, even in the presence of disturbance, model mismatches or reality differences. For example, a wind turbine system must keep its efficiency even in the presence of air velocity changes. In contrast, an example of unwanted disturbance amplification is the Tacoma Narrows suspension bridge, where disturbances in the form of strong winds caused resonant oscillations of increasing magnitude in the bridge structure that ultimately led to its destruction (Buchstaller, 2010).

In particular, the robust state observer design problem is related to finding the observer gains such that it is always possible to estimate the real states within prescribed tolerances, despite the effects of uncertainties. Some types of uncertainties include:

- disturbances effect on the plant
- measurement noise
- modeling mismatches
- gain scheduling mismatches

To deal with this kind of uncertainties, the robust technique called as the H_∞ approach is considered. This approach has been developed since the beginning of the eighties and has been intensely applied with successful results for regular LPV systems (Apkarian et al., 1994; Apkarian and Gahinet, 1995; Masubuchi et al., 1997; Kajiwara et al., 1999).

The H_∞ approach assumes that the inputs belong to a set of norm bounded functions. The idea is to minimize the worst error that can arise from any input in the vector set, and can be expressed as

$$\| u(t) \|_2 = \left(\int_0^\infty u^T(t) u(t) dt \right)^{1/2}. \tag{3.1}$$

Then, for a stable D-LPV system

$$\mathscr{F} := \begin{cases} E\dot{x}(t) & = \sum_{i=1}^{h} \rho_i(\zeta(t)) \{A_i x(t) + B_i u(t)\} \\ y(t) & = Cx(t) \end{cases} \tag{3.2}$$

the H_∞norm is defined as follows:

Definition 3.1.1. (Van der Schaft, 1992) Let $\gamma \geq 0, \gamma \in \mathbb{R}^+$. The system $\mathscr{F}$ is said to have $\mathscr{L}_2$-gain less than or equal to γ if

$$\int_0^\infty \| y(t) \|_2^2 \, d(t) \leq \gamma^2 \int_0^\infty \| u(t) \|_2^2 \, d(t) \tag{3.3}$$

or, equivalently

$$\int_0^\infty y^T(t) y(t) dt \leq \gamma^2 \int_0^\infty u^T(t) u(t) dt \tag{3.4}$$

for all $u(t) \in \mathscr{L}_2[0, \infty]$ and zero initial conditions $x(0) = 0$, with $y(t)$ denoting the output of $\mathscr{F}$ resulting from the input $u(t)$. The system has $\mathscr{L}_2$-gain$< \gamma$ if there exists some $0 \leq \gamma_0 < \gamma$ such that $\mathscr{F}$ holds for γ_0.

Remark 3.1.2. This definition is a generalization of the H_∞ linear attenuation for nonlinear systems. However, when translated into the time domain, the H_∞-norm is nothing else than the $\mathcal{L}_2$-induced norm from the input to the output, commonly called the $\mathcal{L}_2$-gain of $\mathscr{F}$ (Magarotto et al., 1999).

The main challenge in designing state observers with H_∞ performance is to guarantee an asymptotically estimation of the state variables and a prescribed precision for the estimation of a linear combination of the state variables in the presence of disturbances and the error provided by the scheduling functions. To guarantee this objective, three different methods to design state observers with H_∞ performance will be considered in this Chapter. For the sake of simplicity, the different methods are named as follows:

- **Approach 1: Descriptor-observer approach:** A state descriptor-LPV observer for D-LPV systems is proposed.
- **Approach 2: The uncertain system approach**: The D-LPV system is transformed into an uncertain system with estimated gain scheduling functions. Then, a suitable design is proposed by considering the H_∞-norm of the estimation error to guarantee convergence and disturbance attenuation.

- **Approach 3: Uncertain error system approach**: Unlike the second approach, the state estimation error is transformed into an uncertain system by considering the convex property of the scheduling functions. Then, the problem is formulated such that the main target is to minimize the effects of disturbances and the uncertainty given by the scheduling functions on the error system.

For all the approaches, some simulation examples are presented to illustrate the effectiveness of the proposed methods.

3.2 Approach 1: Descriptor observer approach

Let us consider a continuous D-LPV system given by

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i(\zeta(t)) \left[A_i x(t) + B_i u(t)\right] \\ y(t) &= Cx(t) \end{aligned} \tag{3.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ are the state vector, the control input and the measured vector, respectively. A_i, B_i, and C are constant matrices of appropriate dimensions.

Assuming that system (3.5) achieve observability conditions, as detailed in Chapter 2, the following descriptor-LPV observer (D-LPVO) is proposed

$$E\dot{\hat{x}}(t) = \sum_{i=1}^{h} \rho_i(\hat{\zeta}(t)) \left[A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - C\hat{x}(t))\right]$$

$$\hat{y}(t) = C\hat{x}(t) \tag{3.6}$$

$$r(t) = M(y(t) - \hat{y}(t)) \tag{3.7}$$

where $\hat{x}(t)$ represents the estimated state vector. L_i are the gain matrices of appropriate dimensions to be designed. M is a residual weighting matrix to be determined. $r(t)$ is a residual (error) which measures the difference between the real and the estimated outputs[1]. Note that $\rho_i(\hat{\zeta}(t))$ depends on the estimated state $\hat{\zeta}(t)$ and satisfy property (3.69), rewritten here by convenience:

$$\begin{cases} \rho_i(\hat{\zeta}(t)) \geq 0, \forall i \in [1, 2, ..., h], \\ \sum_{i=1}^{h} \rho_i(\hat{\zeta}(t)) = 1, \forall t. \end{cases} \tag{3.8}$$

[1]The error vector $r(t)$, also called the residual, is considered in the observer design as an auxiliary vector to apply the $\mathscr{L}_2$-induced norm from the inputs to the residual in the obtained error system. Later, the residual vector will be considered to perform sensor fault detection and isolation.

The problem is reduced to find the gains L_i of the D-LPV observer to ensure an asymptotic convergence of the estimation error despite the uncertainty provided by the unmeasurable gain scheduling functions.

As discussed in Chapter 2, the main challenge to design a state observer for systems with unmeasurable scheduling functions is to compute the estimation error, in spite of the difference between the scheduling vectors $\zeta(t)$ and $\hat{\zeta}(t)$. To deal with this problem, the proposed approach transforms the D-LPV system (3.5) into an uncertain system with estimated gain scheduling functions as

$$\begin{aligned} E\dot{x}(t) =& \sum_{i=1}^{h} \rho_i(\hat{\zeta}(t)) \left[(A_i + \Delta A)x(t) + (B_i + \Delta B)u(t)\right] \\ y(t) =& Cx(t) \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \Delta A &= \sum_{i=1}^{h} [\rho_i(\zeta(t)) - \rho_i(\hat{\zeta}(t))] A_i = \mathcal{A} F(t) \Phi_A \\ \Delta B &= \sum_{i=1}^{h} [\rho_i(\zeta(t)) - \rho_i(\hat{\zeta}(t))] B_i = \mathcal{B} F(t) \Phi_B \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= [A_1 \dots A_h], \Phi_A = [I_{n_1} \dots I_{n_h}]^T, \\ \mathcal{B} &= [B_1 \dots B_h], \Phi_B = [I_{m_1} \dots I_{m_h}]^T \\ F(t) &= \begin{bmatrix} (\rho_1 - \hat{\rho}_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (\rho_h - \hat{\rho}_h) \end{bmatrix}. \end{aligned}$$

Note that this transformation is possible due to the convex property (3.8), which implies $F^T(t)F(t) \leq I$. The uncertain matrices ΔA and ΔB represent the differences between the estimated and the unmeasurable scheduling functions given as uncertainties.

For convenience in the sequel, ρ_i *and* $\hat{\rho}_j$ *are considered as shorthand notations of* $\rho_i(\zeta(t))$ *and* $\rho_j(\hat{\zeta}(t))$*, respectively.*

The estimation error between the uncertain system (3.9) and the observer (3.6) is given by

$$e(t) = \hat{x}(t) - x(t). \tag{3.10}$$

The estimation error dynamics is obtained as

$$E\dot{e}(t) = E\dot{\hat{x}}(t) - E\dot{x}(t) \tag{3.11}$$

$$E\dot{e}(t) = \sum_{i=1}^{h} \hat{\rho}_i \left[(A_i - L_i C)\hat{x}(t) + B_i u(t) + L_i C x(t)\right) - (A_i + \Delta A)x(t) - (B_i + \Delta B)u(t)]$$

$$E\dot{e}(t) = \sum_{i=1}^{h} \hat{\rho}_i \left[(A_i - L_i C)e(t) - \Delta A x(t) - \Delta B u(t))\right] \tag{3.12}$$

By combining the system (3.5) and the error dynamics (3.12) and by augmenting the states as $x_e(t) = [e(t)^T, x(t)^T]^T$, the residual augmented error dynamics is obtained as follows

$$\mathscr{G}_e := \begin{cases} \bar{E}\dot{x}_e(t) & = \sum_{i=1}^{h} \hat{\rho}_i \sum_{j=1}^{h} \rho_j \left[\bar{A}_{ij} x_e(t) + \bar{B}_{ij} u(t)\right] \\ r(t) & = \bar{C} x_e(t) \end{cases} \tag{3.13}$$

$$\begin{aligned} \bar{E} &= \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \bar{A}_{ij} = \begin{bmatrix} A_i - L_i C & -\Delta A \\ 0 & A_j \end{bmatrix} \\ \bar{B}_{ij} &= \begin{bmatrix} -\Delta B \\ B_j \end{bmatrix}, \bar{C} = \begin{bmatrix} MC & 0 \end{bmatrix} \end{aligned}$$

The problem is reformulated to design the observer gains to guarantee asymptotic convergence to zero despite the uncertainties given by the unmeasurable scheduling parameter. Sufficient conditions to achieve this objective are given through the following theorem:

Theorem 3.2.1. Consider the system (3.5) and the observer (3.6)-(3.7). The estimation error (3.13) is asymptotically stable with attenuation level $\gamma > 0$, such that $\| \mathscr{G}_e \|_\infty < \gamma$, if there exist scalars $\varepsilon_A > 0$, $\varepsilon_B > 0$, Ξ_i, M, and matrices $X = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, such that the following optimization problem holds $\forall i, j \in [1, 2, ..., h]$:

$$\min_{P, Q, \Xi_i, \varepsilon_A, \varepsilon_B} \gamma$$

s.t.

$$E^T P = P^T E \geq 0 \tag{3.14}$$

$$E^T Q = Q^T E \geq 0 \tag{3.15}$$

$$\begin{bmatrix} \mathsf{He}(A_i^T P - C^T \Xi_i^T) & 0 & 0 & P\mathcal{A} & \varepsilon_A \Phi_A^T & P\mathcal{B} & 0 & C^T M^T \\ * & \mathsf{He}(A_j^T Q) & QB_j & 0 & 0 & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 & -\varepsilon_B \Phi_B & 0 \\ * & * & * & -\varepsilon_A I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_A I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_B I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_B I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0. \quad (3.16)$$

The gain matrices of (3.6) are computed by $L_i = P^{-1}\Xi_i$.

Proof. The following lemma will be used to proof Theorem 3.2.1.

Lemma 3.2.1. *(Yang et al., 2005) Let M, H, and Φ be real matrices of appropriate dimensions, with Γ satisfying $\Gamma^T\Gamma \leq I$, then*

$$M + H\Gamma\Phi + \Phi^T\Gamma^T H^T < 0$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$M + \varepsilon\Phi^T\Phi + \frac{1}{\varepsilon}HH^T < 0$$

or, equivalently

$$\begin{bmatrix} M & H & \varepsilon\Phi^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon\Phi & 0 & -\varepsilon I \end{bmatrix} < 0$$

By considering Definition 3.1.1, the residual state error system (3.13) has H_∞ performance, such that $\| \mathcal{G}_e \|_\infty < \gamma$, if

$$J_e := \int_0^\infty \left(r^T(t)r(t) - \gamma^2 u^T(t)u(t) \right) dt < 0 \quad (3.17)$$

and it is also asymptotically stable if there exists a quadratic Lyapunov function

$$V(x_e(t)) = x_e^T(t)\bar{E}^T X x_e(t) < 0$$

with condition

$$\bar{E}^T X = \bar{E}^T X \geq 0 \tag{3.18}$$

J_e becomes

$$J_e := \int_0^\infty \left(r^T(t)r(t) - \gamma^2 u^T(t)u(t) + \dot{V}(x_e(t))\right) dt - V(x_e(t)) < 0 \tag{3.19}$$

$$= \int_0^\infty \left(x_e^T(t)\bar{C}^T\bar{C}x_e(t) - \gamma^2 u(t)^T u(t) - (\bar{E}\dot{x}_e^T(t))Xx_e(t) - x_e^T(t)X^T\bar{E}\dot{x}_e(t)\right) d(t) - V(x_e(t) < 0 \tag{3.20}$$

$$= \int_0^\infty \left(\begin{bmatrix} x_e^T(t) \\ u^T(t) \end{bmatrix}^T \begin{bmatrix} \bar{C}^T\bar{C} & 0 \\ * & -\gamma^2 \end{bmatrix} \begin{bmatrix} x_e(t) \\ u(t) \end{bmatrix} - \sum_{i,j=1}^{h} \hat{\rho}_i \rho_i \left(x_e(t)^T \bar{A}_{ij}^T + u^T(t)\bar{B}_{ij}^T\right) X x_e(t) \right.$$

$$\left. + x_e^T(t)X^T \sum_{i,j=1}^{h} \hat{\rho}_i \rho_i \left(\bar{A}_{ij}x_e(t) + \bar{B}_{ij}u(t)\right) \right) - V(x_e(t)) < 0 \tag{3.21}$$

$$J_e := \int_0^\infty \sum_{i,j=1}^{h} \hat{\rho}_i \rho_i \left(\begin{bmatrix} x_e^T(t) \\ u^T(t) \end{bmatrix}^T \begin{bmatrix} \bar{A}_{ij}^T X + X^T \bar{A}_{ij}^T + \bar{C}^T\bar{C} & X^T\bar{B}_{ij} \\ * & -\gamma^2 \end{bmatrix} \begin{bmatrix} x_e(t) \\ u(t) \end{bmatrix} \right) - V(x_e(t)) < 0 \tag{3.22}$$

Then, by considering the Schur complement, the inequality (3.22) is rewritten in the LMI equivalent form

$$\bar{E}^T X = X^T \bar{E} \quad > \quad 0 \tag{3.23}$$

$$\begin{bmatrix} \bar{A}^T X + P^T \bar{A} & X^T \bar{B} & \bar{C}^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \quad < \quad 0. \tag{3.24}$$

where $X = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$. Considering the augmented matrices in (3.13), the LMI system (3.24) becomes

$$\begin{bmatrix} \mathsf{He}(A_i^T P - C^T \Xi_i^T) & -P\Delta A & -P\Delta B & C^T M^T \\ * & \mathsf{He}(A_j^T Q) & QB_j & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0. \tag{3.25}$$

Note that the previous equation can be rewritten equivalently as

$$\overbrace{\begin{bmatrix} \mathsf{He}(A_i^T P - C^T \Xi_i^T) & 0 & 0 & C^T M^T \\ * & \mathsf{He}(A_j^T Q) & QB_j & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix}}^{\mathscr{M}} + \overbrace{\begin{bmatrix} 0 & -P\Delta A & -P\Delta B & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}}^{\mathcal{H}} < 0.$$

Additionally, $\mathcal{H}$ can be rewritten as

$$\mathcal{H} = \mathsf{He}(\tilde{H}_A F_A \tilde{\Phi}_A) + \mathsf{He}(\tilde{H}_B F_B \tilde{\Phi}_B) \tag{3.26}$$

with

$$\begin{aligned} \tilde{H}_A F_A \tilde{\Phi}_A &= \begin{bmatrix} P\mathcal{A} \\ 0 \\ 0 \\ 0 \end{bmatrix} F(t) \begin{bmatrix} 0 & \Phi_A & 0 & 0 \end{bmatrix} \\ \tilde{H}_B F_B \tilde{\Phi}_B &= \begin{bmatrix} P\mathcal{B} \\ 0 \\ 0 \\ 0 \end{bmatrix} F(t) \begin{bmatrix} 0 & 0 & \Phi_B & 0 \end{bmatrix} \end{aligned}$$

By considering the above nomenclature, (3.25) can be rewritten as

$$\mathcal{M} + \mathsf{He}(\tilde{H}_A F_A \tilde{\Phi}_A) + \mathsf{He}(\tilde{H}_B F_B \tilde{\Phi}_B) \tag{3.27}$$

Using the Lemma 3.2.1, the following expression is obtained

$$\begin{bmatrix} \mathsf{He}(A_i^T P - C^T \Xi_i^T) & 0 & 0 & C^T M^T & P\mathcal{A} & 0 & P\mathcal{B} & 0 \\ * & \mathsf{He}(A_j^T Q) & QB_j & 0 & 0 & \varepsilon_A \Phi_A^T & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & -\varepsilon_B \Phi_B \\ * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_A I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_A I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_B I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_B I \end{bmatrix} < 0 \tag{3.28}$$

In order to rearrange columns 3 and 8 from (3.28), the LMI (3.28) is pre- and post-multiplied by

$$\mathscr{I} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$$

and its transpose to obtain the LMI (3.16). Finally, by considering (3.23), the LMI given in (3.14) is obtained. This completes the proof. □

It should be noticed that, after verifying observability of (3.5) and solving (3.16), the robust stability of (3.13) is guarantee. Nevertheless, due to the equality constraint (3.14), the LMI system could become numerically infeasible due to the strict condition given by the equality (3.14). The following Lemma provides sufficient conditions in order to avoid numerical problems or infeasible solutions, by eliminating the equality constraint $\bar{E}^T X = \bar{E}^T X$.

Lemma 3.2.2. *(Xu and Lam, 2006) All $Z \in \mathbb{R}^{n\times n}$ satisfying*

$$E^T Z = Z^T E \geq 0$$

can be parametrized as

$$\mathscr{Z} = ZE + SX$$

where $Z > 0 \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{(n-r)\times n}$ are parameter matrices. $S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^T S = 0$, where $S \in \mathbb{R}^{n\times(n-r)}$ is any full column rank matrix and satisfies $E^T S = 0$.

Corollary 3.2.1. Consider the system (3.5) and the observer (3.6)-(3.7). The estimation error (3.13) is asymptotically stable with attenuation level $\gamma > 0$, such that $\| \mathscr{G}_e \|_\infty < \gamma$, if there exist scalars $\varepsilon_A > 0$, $\varepsilon_B > 0$, matrices X_1, $\mathscr{P} = PE + SX_1$, $P > 0$, Ξ_i, M, X_2, and $\mathscr{Q} = QE + SX_2$, $Q > 0$, such that the following optimization problem holds $\forall i, j \in [1, 2, ..., h]$:

$$\min_{P, Q, \Xi_i, X_1, X_2\, \varepsilon_A, \varepsilon_B} \gamma$$

$$\text{s.t.}$$

$$\begin{bmatrix} \mathsf{He}(A_i^T\mathscr{P}-C^T\Xi_i^T) & 0 & 0 & \mathscr{P}\mathcal{A} & \varepsilon_A\Phi_A^T & \mathscr{P}\mathcal{B} & 0 & C^TM^T \\ * & \mathsf{He}(A_j^T\mathscr{Q}) & \mathscr{Q}B_j & 0 & 0 & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 & -\varepsilon_B\Phi_B & 0 \\ * & * & * & -\varepsilon_A I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_A I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_B I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_B I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \quad (3.29)$$

where S any full column rank matrix and satisfies $E^TS=0$.

The gain matrices are computed as $L_i = \mathscr{P}^{-1}\Xi_i$.

Proof. Sufficient condition for the LMI set (3.29) are obtained by substituting $\mathscr{P}_2 = P_2E+SX_1$ and $\mathscr{Q} = QE+SX_2$, and with condition $E^TS=0$ is easy to prove that $\bar{E}^TX = X^T\bar{E} \geq 0$. □

Remark 3.2.1. The LMI set (3.29) can brings conservatism into the observer design due to the LMI dimension, the number of models, and the requirement of a single matrix P and Q. To reduce the conservatism, relaxed conditions can be obtained by considering Lemma B.2.17. Then, without loss of good compromise between complexity and conservatism, the following LMIs set can be considered:

$$\Upsilon_{ii} < 0 \quad (3.30)$$

$$\frac{2}{h-1}\Upsilon_{ii}+\Upsilon_{ij}+\Upsilon_{ji} \leq 0 \quad (3.31)$$

where Υ_{ij} is given by (3.29).

Example 3.2.2. In this section, a numerical example is presented to illustrate the proposed method. Consider the D-LPV system form (3.5) defined by the following matrices

$$E = \mathsf{diag}(1,1,0), A_1 = \begin{bmatrix} -10 & 5 & 6.5 \\ 2 & -5.5 & -1.25 \\ -9 & 4 & 8.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -10 & 5 & 6.5 \\ 5 & -4 & -1.25 \\ -2 & 4 & 7 \end{bmatrix}, A_3 = \begin{bmatrix} -8 & 5 & 6.5 \\ 5 & -4 & -1.25 \\ -5 & 4 & 6 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.7 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 0.5 \\ 0.6 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Convex scheduling functions, depending on the unmeasurable scheduling vector $x(t)$, are defined as:

$$\begin{aligned} \rho_i(x(t)) &= \frac{\mu_i(x(t))}{\sum_{i=1}^{3} \mu_i(x(t))}, \forall i \in [1,2,3], \\ \mu_1(x(t)) &= exp\left[\frac{1}{2}\left(\frac{x_1(t)+0.4}{0.5}\right)^2\right] \\ \mu_2(x(t)) &= exp\left[\frac{1}{2}\left(\frac{x_1(t)-0.4}{0.1}\right)^2\right] \\ \mu_3(x(t)) &= exp\left[\frac{1}{2}\left(\frac{x_1(t)-1}{0.5}\right)^2\right]. \end{aligned} \tag{3.32}$$

The gains of the descriptor state observer (3.6)-(3.7), were computed according to Corollary 3.2.1 (LMI 3.29) using the Yalmip Toolbox (Lofberg, 2004). The following matrices obtained:

$$P = P^T = \begin{bmatrix} 0.0062 & 0.0031 & 0.0000 \\ 0.0031 & 0.0059 & 0.0000 \\ 0.0000 & 0.0000 & 66.4719 \end{bmatrix} > 0, S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, X = 1 \times 10^{-4} \begin{bmatrix} -0.5364 & -0.1341 & 0 \end{bmatrix}$$

$$\Xi_1 = \begin{bmatrix} 0.1706 & -0.0000 & 0.0000 \\ -0.0005 & 0.1708 & -0.3370 \\ -0.0001 & 0.1662 & -0.0000 \end{bmatrix}, \Xi_2 = \begin{bmatrix} 0.1704 & -0.0000 & -0.0000 \\ -0.0004 & 0.1709 & -0.3371 \\ -0.0000 & 0.1662 & -0.0000 \end{bmatrix},$$

$$\Xi_3 = \begin{bmatrix} 0.1706 & 0.0000 & -0.0000 \\ -0.0003 & 0.1709 & -0.3371 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}, Q = Q^T = \begin{bmatrix} 0.4719 & 0.0236 & -0.0000 \\ * & 0.0667 & -0.0000 \\ * & * & 0.6647 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} -0.3849 & -0.0479 & -0.2892 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, by considering $L_i = (PE + SX_1)^{-1}\Xi_i$, the gain matrices are given as follows

$$L_1 = \begin{bmatrix} 37.2421 & -19.6149 & 38.6931 \\ -19.7055 & 39.0773 & -77.0854 \\ -0.0000 & 0.0025 & 0.0000 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} 37.1657 & -19.6251 & 38.7010 \\ -19.6431 & 39.0977 & -77.1012 \\ -0.0000 & 0.0025 & 0.0000 \end{bmatrix},$$
$$L_3 = \begin{bmatrix} 37.2161 & -19.6251 & 38.6994 \\ -19.6524 & 39.0977 & -77.0982 \\ 0.0000 & 0.0025 & 0.0000 \end{bmatrix},$$

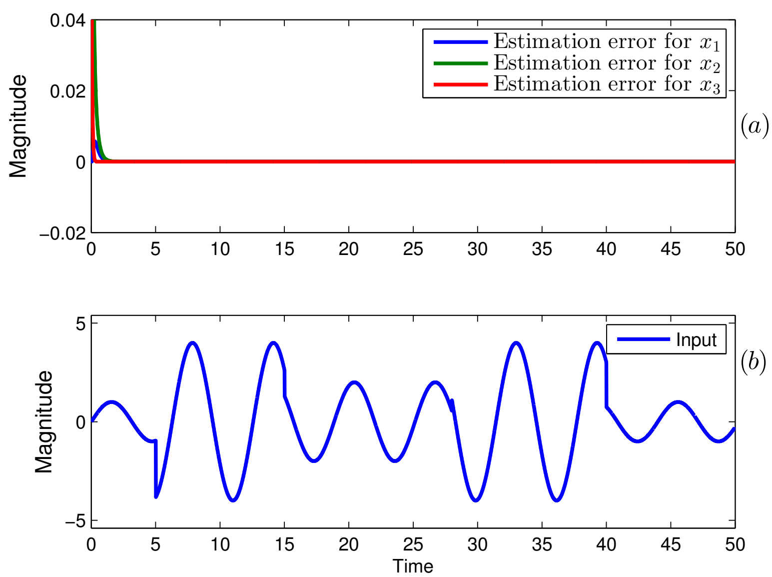


Figure 3.1 – Observer performance: (a) Output estimation error, (b) input.

The gain matrices are close, but not the same, due to the fact that Theorem 3.2.1 computes a common P matrix for all the local models. Nevertheless, the simulations will show that the observer performance is not affected. The computed attenuation level is $\gamma = 0.0323$. The simulation is done by considering initial conditions $x(0) = [0.1, 1, -1]^T$ and $\hat{x}(0) = [1, 0.5, 0]^T$ for the system and the observer, respectively. Fig. 3.1b shows the applied input signal. The quadratic state estimation error is displayed in Fig. 3.1a. Since the input is bounded by 4 and with computed $\gamma = 0.312$, the estimation error is bounded by 1.24.

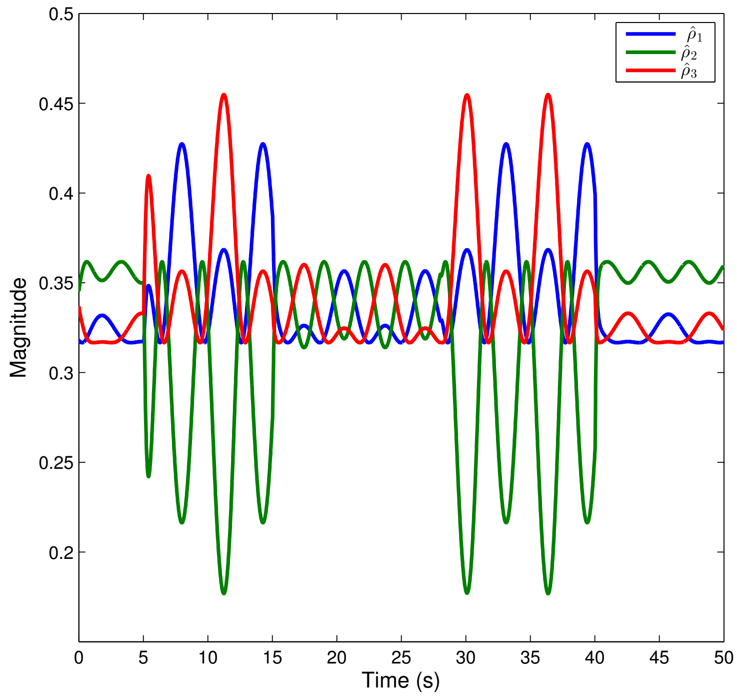


Figure 3.2 – Estimated scheduling functions $\rho_i(\hat{x}(t))$.

The magnitude of the bounded error is small considering the magnitude of the system response. The estimated gain scheduling functions are displayed in Fig. 3.2. It should be noted that the evolution of the scheduling functions shows the interaction and weight of the three models to the global dynamics during the simulation time.

This method reduces some conservatism by considering less gain matrices compared with the standard observer, which results in LMIs with less dimensions. For example, for a D-LPV system of three models, only three gains matrices are computed compared with the eleven necessary in the other approaches. Nevertheless, this approach does not consider disturbance rejection, this problem will be addressed in future research. Also, as detailed in Chapter 2, the observers designed in descriptor form are sensitive to high-frequency noise, which can reduce its applicability and performance. The following approach solves this issue by considering a standard observer with disturbance and noise attenuation.

3.3 Approach 2: The uncertain system approach

Consider the following continuous D-LPV system affected by disturbances as:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i \left[A_i x(t) + B_i u(t) + B_{di} d(t)\right] \\ y(t) &= Cx(t) + D_d d(t) \end{aligned} \tag{3.33}$$

where $d \in \mathbb{R}^q$ is the disturbance/noise vector. B_{di} and D_d are constant matrices of appropriate dimensions.

Then, for a D-LPV system (3.33), R- and I-observable (2.37)-(2.38), the following robust state observer is proposed:

$$\dot{z}(t) = \sum_{j=1}^{h} \hat{\rho}_j \left[N_j z(t) + G_j u(t) + L_j y(t)\right]$$

$$\hat{x}(t) = z(t) + T_2 y(t) \tag{3.34}$$

$$r(t) = M(y(t) - C\hat{x}(t)) \tag{3.35}$$

where $z(t) \in \mathbb{R}^n$ represents the state vector of the observer, $\hat{x}(t)$ the estimated state vector. N_j, G_j, L_j, and T_2 are the gain matrices of appropriate dimensions to be synthesized. $r(t)$ is the residual vector that measures the error between the estimated and the measured outputs. M is a residual weighting matrix to be determined. The estimation problem is reduced to synthesize the gain matrices N_j, G_j, L_j, and T_2, $\forall i \in [1, 2, \ldots, h]$, to guarantee the convergence of the state estimation error and to maximize the robustness against unmeasurable scheduling parameters $\hat{\rho}_j$ and disturbances $d(t)$.

The state-space error equation is obtained by connecting system (3.33) and observer (3.34). However, due to the fact that (3.33) and (3.34) are depending on different scheduling vector, $x(t)$ and $\hat{x}(t)$, the state-space error should be affected by such difference. To deal with this problem, the system (3.34) is transformed into an uncertain system with estimated gain scheduling function by adding and subtracting the term $\sum_{j=1}^{h} \hat{\rho}_j \left(A_j x(t) + B_j u(t)\right)$ as

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i \left[A_i x(t) + B_i u(t) + B_{di} d(t)\right] - \sum_{j=1}^{h} \hat{\rho}_j \left(A_j x(t) + B_j u(t)\right) + \sum_{j=1}^{h} \hat{\rho}_j \left(A_j x(t) + B_j u(t)\right) \\ &= \sum_{j=1}^{h} \hat{\rho}_j \left(A_j x(t) + B_j u(t)\right) + \left(\sum_{i=1}^{h} \rho_i A_i - \sum_{j=1}^{h} \hat{\rho}_j A_j\right) x(t) + \left(\sum_{i=1}^{h} \rho_i B_i - \sum_{j=1}^{h} \hat{\rho}_j B_j\right) + B_{di} d(t) \end{aligned}$$

Then, the system (3.33) becomes

$$E\dot{x}(t) = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left[\tilde{A}_{ij}x(t) + \tilde{B}_{ij}u(t) + B_{di}d(t)\right]$$
$$y(t) = Cx(t) + D_d d(t) \tag{3.36}$$

with

$$\sum_{i,j=1}^{h} \rho_i \hat{\rho}_j = \sum_{i=1}^{h}\sum_{j=1}^{h} \rho_i(\zeta(t))\rho_j(\hat{\zeta}(t)),\ \forall i,j \in [1,2,...,h],$$
$$\tilde{A}_{ij} = A_j + \Delta A_{ij},\ \Delta A_{ij} = A_i - A_j,$$
$$\tilde{B_{ij}} = B_j + \Delta B_{ij},\ \Delta B_{ij} = B_i - B_j.$$

The state-space estimation error is defined as:

$$e(t) = x(t) - \hat{x}(t) \tag{3.37}$$
$$e(t) = (I - T_2 C)x(t) - z(t) - T_2 D_d d(t), \tag{3.38}$$

assuming that there exists a matrix $T_1 \in \mathbb{R}^{n\times n}$ such that

$$T_1 E = I - T_2 C. \tag{3.39}$$

In order to eliminate the influence of $d(t)$ in (3.38), the following is assumed

$$T_2 D_d = 0, \tag{3.40}$$

together with (3.39), the following relationship is obtained

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} E & 0 \\ C & D_d \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix}. \tag{3.41}$$

Then, a particular solution of matrices T_1 and T_2 is computed as

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ C & D_d \end{bmatrix}^{\dagger}. \tag{3.42}$$

By considering previous manipulations, the error becomes

$$e(t) = T_1 E x(t) - z(t) \tag{3.43}$$

The state-space error equation is given by

$$\begin{aligned}
\dot{e}(t) &= T_1 E\dot{x}(t) - \dot{z}(t) && (3.44)\\
\dot{e}(t) &= \sum_{i,j=1}^{h} \rho_i\hat{\rho}_j \left[T_1\left(\tilde{A}_{ij}x(t)+\tilde{B}_{ij}u(t)+B_{di}d(t)\right) - \left(N_j z(t)+G_j u(t)+L_j y(t)\right)\right]\\
\dot{e}(t) &= \sum_{i,j=1}^{h} \rho_i\hat{\rho}_j \left[\left(T_1A_j - L_jC - N_jT_1E\right)x(t) + T_1\Delta A_{ij}x(t) + \left(T_1B_j - G_j\right)u(t) + T_1\Delta B_{ij}u(t)\right.\\
&\quad \left. + \left(T_1B_{di} + N_jT_2D_d - L_jD_d\right)d(t) + N_je(t)\right]. && (3.45)
\end{aligned}$$

The following assumptions are considered

$$\begin{aligned}
T_1A_j - L_jC - N_jT_1E &= 0 && (3.46)\\
G_j &= T_1B_j. && (3.47)
\end{aligned}$$

Manipulating (3.46), the gain synthesis is obtained as

$$N_j = T_1A_j + K_jC \tag{3.48}$$

$$K_j = N_jT_2 - L_j. \tag{3.49}$$

Substituting these conditions on (3.45) and considering (3.35), the state-space residual equation is obtained as

$$\begin{aligned}
\dot{e}(t) &= \sum_{i,j=1}^{h} \rho_i\hat{\rho}_j \left[N_je(t) + T_1\Delta A_{ij}x(t) + T_1\Delta B_{ij}u(t) + \left(T_1B_{di} + K_jD_d\right)d(t)\right] && (3.50)\\
r(t) &= M(y(t) - C\hat{x}(t)) = MCe(t) + MD_dd(t).
\end{aligned}$$

In order to construct a state-space error system, the states are extended as $x_e(t) = \begin{bmatrix} e(t)^T & x(t)^T \end{bmatrix}^T$. Then, the residual state-space error system is rewritten in compact form as

$$\begin{aligned}
\bar{E}\dot{x}_e(t) &= \bar{A}x_e(t) + \bar{B}_d\bar{d}(t)\\
r(t) &= \bar{C}x_e(t) + \bar{D}_d\bar{d}(t) && (3.51)
\end{aligned}$$

with

$$\bar{E} = \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \bar{A} = \sum_{i,j=1}^{h} \rho_i\hat{\rho}_j \begin{bmatrix} N_j & T_1\Delta A_{ij} \\ 0 & A_i \end{bmatrix},$$

$$\bar{B}_d = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \begin{bmatrix} T_1 \Delta B_{ij} & T_1 B_{di} + K_j D_d \\ B_i & B_{di} \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} MC & 0 \end{bmatrix}, \bar{D}_d = \begin{bmatrix} 0 & MD_d \end{bmatrix}, \bar{d}(t) = \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}.$$

The observer gains of (3.34) are synthesized by considering the augmented error equation (3.51). The error should converge asymptotically to zero despite the disturbances and the unmeasurable scheduling function. In order to achieve this goal, the problem is reformulated in the H_∞ criterion. The following Theorem provides sufficient conditions to compute the observer gains:

Theorem 3.3.1. Given system (3.33), the observer (3.34) and let the attenuation level $\gamma > 0$; the residual state-space error system (3.51) is asymptotically stable with H_∞ performance if it satisfies $\| r(t) \|_2^2 \leq \gamma^2 \| \bar{d}(t) \|_2^2$ and if there exist matrices $P_1 = P^T > 0$, M, Q_j, and P_2, $\forall i, j \in [1, 2, ..., h]$, such that:

$$E^T P_2 = P_2^T E \geq 0 \tag{3.52}$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & P_1 T_1 \Delta B_{ij} & \Xi_{14} & C^T M^T \\ * & \Xi_{22} & P_2 B_i & P_2 B_{di} & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\gamma^2 I & D_d^T M^T \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{3.53}$$

with:

$$\Xi_{11} = (T_1 A_j)^T P_1 + P_1 (T_1 A_j) + (Q_j C)^T + Q_j C,$$
$$\Xi_{12} = P_1 T_1 \Delta A_{ij},$$
$$\Xi_{14} = P_1 T_1 B_d + Q_j D_d,$$
$$\Xi_{22} = A_i^T P_2 + P_2^T A_i,$$

where T_1 is given by

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ C & D_d \end{bmatrix}^{\dagger}.$$

Then, the gain matrices of observer (3.34) are given by $K_j = P_1^{-1} Q_j$ and the equations defined in (3.47)-(3.49).

Proof. As defined in Section 3.1, the H_∞ norm of (3.51) can be written as the $\mathscr{L}_2$-induced norm form $r(t)$ to the $d(t)$ as:

$$J_{r\bar{d}} := \| r(t) \|_2^2 \leq \gamma^2 \| \bar{d}(t) \|_2^2 . \tag{3.54}$$

Equivalently, $J_{r\bar{d}}$ is rewritten as

$$J_{r\bar{d}} := \int_0^\infty r^T(t) r(t) dt - \gamma^2 \int_0^\infty \bar{d}^T(t) \bar{d}(t) dt < 0 \tag{3.55}$$

in order to represents the $\mathscr{L}_2$ gain of system (3.51) from $\bar{d}(t)$ to $r(t)$ bounded by γ. A Lyapunov function defined as $\Omega(t) = V(x_e(t)) = x_e^T(t) \bar{E}^T P x_e(t)$ is considered to guarantee stability, such that $J_{r\bar{d}}$ is manipulated as

$$\begin{aligned} J_{r\bar{d}} : &= \int_0^\infty \left(r^T(t) r(t) - \gamma^2 \bar{d}^T(t) \bar{d}(t) + \dot{\Omega}(t) \right) dt - \Omega(t) < 0 \\ J_{r\bar{d}} : &= \int_0^\infty \left((\bar{C} x_e(t) + \bar{D}_d \bar{d}(t))^T (\bar{C} x_e(t) + \bar{D}_d \bar{d}(t)) - \gamma^2 \bar{d}^T(t) \bar{d}(t) + \dot{V}(x_e(t)) \right) d\tau - \Omega(t) < 0 \end{aligned}$$

$$\begin{aligned} J_{r\bar{d}} := \int_0^\infty \Big(& x_e^T(t) \left(\bar{C}^T \bar{C} \right) x_e(t) + x_{e^T}(t) \left(\bar{C}^T \bar{D}_d \right) \bar{d}(t) + \bar{d}^T(t) \left(\bar{D}_d{}^T \bar{C} \right) x_e(t) \\ & + \bar{d}^T(t) \left(\bar{D}_d{}^T \bar{D}_d \right) \bar{d}(t) - \gamma^2 \bar{d}^T(t) \bar{d}(t) + \dot{\Omega}(t) \Big) dt - \Omega(t) < 0 \end{aligned}$$

$$\begin{aligned} J_{r\bar{d}} : &= \int_0^\infty \left(\begin{bmatrix} e(t) \\ x(t) \\ \bar{d}(t) \end{bmatrix}^T \begin{bmatrix} \bar{C}^T \bar{C} & 0 & \bar{C}^T \bar{D}_d \\ * & 0 & 0 \\ * & * & \bar{D}_d^T \bar{D}_d - \gamma^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \\ \bar{d}(t) \end{bmatrix} + \dot{\Omega}(t) \right) dt - \Omega(t) < 0 \\ J_{r\bar{d}} &= \int_0^\infty \left(\begin{bmatrix} e(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} (MC)^T MC & 0 & 0 & (MC)^T MD_d \\ * & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & (MD_d^T) MD_d - \gamma^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix} + \dot{\Omega}(t) \right) dt - \Omega(t) < 0 \end{aligned} \tag{3.56}$$

The dynamics of the Lyapunov equation, under the assumption $\bar{E}^T P = P^T \bar{E}$, is given by $\dot{\Omega}(t) = \dot{V}(x_e(t)) =$

$\dot{x}_e^T(t)\bar{E}^T P x_e(t) + x_e^T \bar{E}^T P \dot{x}_e(t)$, such that

$$\dot{\Omega}(t) = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left[x_e^T(t)\bar{A}^T P x_e(t) + \bar{d}^T(t)\bar{B}_d^T P x_e(t) + x_e^T(t) P^T \bar{A} x_e(t) + x_e^T(t) P^T \bar{B}_d \bar{d}(t)\right]$$
$$= \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \begin{bmatrix} x_e(t) \\ \bar{d}(t) \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P + P^T \bar{A} & P^T \bar{B}_d \\ * & 0 \end{bmatrix} \begin{bmatrix} x_e(t) \\ \bar{d}(t) \end{bmatrix}$$

with $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, and $\bar{E}^T P = P^T \bar{E} \geq 0$, this inequality is manipulated as

$$\begin{bmatrix} P_1 & 0 \\ 0 & E^T P_2 \end{bmatrix} = \begin{bmatrix} P_1^T & 0 \\ 0 & P_2^T E \end{bmatrix} \geq 0. \tag{3.57}$$

It can be seen that $P_1 = P_1^T \geq 0$ and $E^T P_2 = P_2^T E \geq 0$. Also, $\bar{A}^T P + P^T \bar{A} < 0$. It is equivalent to

$$\begin{bmatrix} \Xi_{11} & P_1 T_1 \Delta A_{ij} \\ * & A_i^T P_2 + P_2^T A_i \end{bmatrix} < 0. \tag{3.58}$$

By considering the extended matrices given in (3.51) the dynamics of the Lyapunov equation Ω becomes

$$\dot{\Omega}(t) = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left\{ \begin{bmatrix} e(t) \\ x(t) \\ \bar{d}(t) \end{bmatrix}^T \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & P_2 B_d \\ * & * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \\ \bar{d}(t) \end{bmatrix} \right\}$$

$$\dot{\Omega}(t) = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left\{ \begin{bmatrix} e(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} \Xi_{11} & \Xi_{12} & P_1 T_1 \Delta B_{ij} & \Xi_{14} \\ * & \Xi_{22} & P_2 B_i & P_2 B_{di} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix} \right\} \tag{3.59}$$

Together (3.56)-(3.59), as defined in (3.55), are rearranged as:

$$J_{r\bar{d}} := \int_0^\infty \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left\{ \begin{bmatrix} e(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix}^T \Theta \begin{bmatrix} e(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix} \right\} < 0 \tag{3.60}$$

where

$$\Theta = \begin{bmatrix} \Xi_{11} + (MC)^T MC & \Xi_{12} & P_1 T_1 \Delta B_{ij} & (MC)^T MD_d \\ * & \Xi_{22} & P_2 B_i & P_2 B_{di} \\ * & * & -\gamma^2 I & 0 \\ * & * & * & (MD_d)^T MD_d - \gamma^2 I \end{bmatrix} < 0. \tag{3.61}$$

it can be seen that if $\Theta < 0$, then $J_{r\bar{d}} < 0$. Finally, the Schur complement implies (3.53). This completes the proof. □

As the previous Approach, the solution of Theorem 3.3.1 could reach to an infeasible solution due to the equality (3.52). Then, by considering Lemma 3.2.2, the following Corollary of Theorem 3.3.1 is obtained:

Corollary 3.3.1. Given system (3.33), the observer (3.34) and let the attenuation level $\gamma > 0$. The residual state-space error system (3.51) is asymptotically stable with H_∞ performance if it satisfies $\| r(t) \|_2^2 \leq \gamma^2 \| \bar{d}(t) \|_2^2$ and if there exist matrices $P_1 = P_1^T > 0$, Q_j, X, and $\mathscr{P}_2 = P_2 E + SX$, $P_2 > 0$, $\forall i, j \in [1, 2, ..., h]$, such that

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & P_1 T_1 \Delta B_{ij} & \Xi_{14} & C^T M^T \\ * & \Xi_{22} & \mathscr{P}_2 B_i & \mathscr{P} B_{di} & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\gamma^2 I & D_d^T M^T \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{3.62}$$

with

$$\Xi_{22} = A_i^T \mathscr{P}_2 + \mathscr{P}_2^T A_i,$$

where T_1 is given by

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ C & D_d \end{bmatrix}^\dagger.$$

$S \in \mathbb{R}^{n \times (n-r)}$ is any full column rank matrix and satisfies $E^T S = 0$.

Then, the gain matrices of observer (3.34) are computed by $K_j = P_1^{-1} Q_j$ and the equations given in (3.47)-(3.49).

The main advantage of this approach is its simplicity, from the conceptual and the numerical point of view. The derived Theorem 3.3.1 and the respectively Corollary 3.3.1 guarantee the asymptotic stability of the error system (3.51), despite the disturbance and the error provided by the unmeasurable scheduling function. However, the method can be improved by considering an uncertain system, which considers the convex property of scheduling functions, but the difficulty of the design is also increased. A solution to this problem will be addressed in the next section.

Example 3.3.1. The proposed method is evaluated via numerical simulations by using a fourth-order mathematical model of an anaerobic bioreactor, which has been previously described and validated in Martínez-Sibaja et al. (2011). This model represents an up-flow anaerobic sludge blanket bioreactor (UASB). The state variables are: $x_1(t) = x_o(t)$ the concentration of the anaerobic biomass; $x_2(t) = s_1(t)$, the concentration of organic matter expressed as chemical oxygen demand (COD); $x_3(t) = Q_{CH4}(t)$ the outlet flow of methane bio-gas, and $x_4(t) = \mu(t)$ the specific growth rate. The input variables are: $u_1(t) = D(t)$ the dilution rate and $u_2(t) = s_1^i(t)$ the concentration of COD in the yield affluent. From the model given in (Martínez-Sibaja et al., 2011), the following nonlinear descriptor system is deduced

$$\dot{x}(t) = f_1(x(t), u(t)) \tag{3.63}$$

$$0 = f_2(x(t), u(t)) \tag{3.64}$$

with

$$f_1(x(t), u(t)) = \begin{bmatrix} Y_1\mu(t)x_o(t) - \alpha D(t)x_o(t) - k_d x_o(t) \\ D(t)(s_1(t)^i - s_1(t)) - \mu(t)x_o \\ (1 - Y_1)Y_{CH4}\mu(t)x_o - Q_{CH4}(t) \end{bmatrix} \tag{3.65}$$

$$f_2(x(t), u(t)) = k_{m1}\frac{s_1(t)}{k_{s1} + s_1(t)} I_{pH} - \mu(t) \tag{3.66}$$

Table 3.1 – Model parameters

Parameter	Value
k_{m1}	5.1 gCOD/gCOD d
k_{s1}	0.5 gCOD/l
k_d	0.02 l/d
Y_1	0.1gCOD/g COD
Y_{CH4}	0.35 l_{CH4}/g COD
α	0.5
I_{pH}	0.9068

where k_{ml}, k_d, k_{s1} are the specific growth rates of mass, the dilution rate of the anaerobic reactor and the constant decrease of semi-saturation for the biomass, respectively. Y_1 is the yield coefficient of degradation of COD, I_{pH} represents the pH inhibition, where pH_{LL} and pH_{UL} are the lower and higher pH limits. The values of the constants parameters are shown in Table 3.1. The process outputs are $y_1(t) = x_o$, $y_2(t) = s_1(t)$ and $y_3(t) = Q_{CH4}(t)$.

The nonlinear descriptor system given by (3.65)-(3.66) can be written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ 0 \end{bmatrix} = \begin{bmatrix} -k_d & 0 & 0 & Y_1 x_1(t) \\ 0 & 0 & 0 & -x_1(t) \\ 0 & 0 & -1 & (1-Y_1)Y_{CH4}x_1(t) \\ 0 & \frac{k_{m1}}{k_{s1}+\tilde{x}_2(t)} I_{pH} & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} -\alpha x_1(t) & 0 \\ -x_2(t) & u_1(t) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \tag{3.67}$$

In order to obtain a D-LPV representation, the scheduling variable are chosen as $\underline{a}_k < \zeta(t) < \bar{a}_k, \forall k = 1,2...4$, where k is the number of non-constants elements in system (3.67). The minimum and maximum bounded limits, $\underline{a}_k$ and $\bar{a}_k$, $k = 1,...,4$, are selected according to experimental constraints, as given in Martínez-Sibaja et al. (2011):

$$\begin{aligned} \zeta_1 &= x_1(t) \in [\underline{a}_1, \bar{a}_1] = [0.2\ 0.7], \\ \zeta_2 &= x_2(t) \in [\underline{a}_2, \bar{a}_2] = [0.01\ 0.6], \\ \zeta_3 &= u_1(t) \in [\underline{a}_3, \bar{a}_3] = [0.2\ 0.8], \\ \zeta_4 &= \frac{k_{m1}}{k_{s1}+x_2(t)} I_{pH} \in [\underline{a}_4, \bar{a}_4] = [3.7\ 9.9]. \end{aligned}$$

For each ζ_j, two local scheduling functions are constructed as

$$\mu_1^k(\zeta_k) = \frac{\bar{a}_k - \zeta_k}{\bar{a}_k - \underline{a}_k}, \mu_2^k = 1 - \mu_1^k, k = 1,2...4. \tag{3.68}$$

Therefore, for $k = 4$, $2^4 = 16$ scheduling functions are computed as the product of the weighting functions that correspond to each local model

$$\rho_i(\zeta(t)) = \prod_{k=1}^{4} \mu_{ik}(\zeta_i) \tag{3.69}$$

Note that the scheduling functions are normal $\rho_i(\zeta(t)) \geq 0, \sum_{i=1}^{16} \rho_i(\zeta(t)) = 1$.

By considering the scheduling variables on the nonlinear matrix (3.67), a descriptor quasi-LPV model is obtained as

$$
\begin{aligned}
E\dot{x}(t) &= \sum_{i=1}^{16} \rho_i(\zeta(t))\,[A_i x(t) + B_i u(t)] \\
y(t) &= Cx(t)
\end{aligned}
\tag{3.70}
$$

$$
\begin{aligned}
E &= \text{diag}(1,1,1,0), \\
A_i &= \begin{bmatrix} -k_d & 0 & 0 & Y_1\zeta_1 \\ 0 & 0 & 0 & -\zeta_1 \\ 0 & 0 & -1 & (1-Y_1)Y_{CH4}\zeta_1 \\ 0 & \zeta_4 & 0 & -1 \end{bmatrix}, B_i = \begin{bmatrix} -\alpha\zeta_1 & 0 \\ -\zeta_2 & \zeta_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Note that the matrices of (3.70) are not unique and different representations can be obtained by considering different arrangements of the scheduling variables. The 16 models of (3.70) are given by evaluating A_i and B_i over the operation ranges of ζ_k. State matrices are not displayed here due to space limitations.

To validate the D-LPV model the following conditions are considered: $x(0) = [0.523\ 0.345\ 0.00001\ 0.001]^T$. The simulated inputs are shown in Fig. 3.3.

Fig 3.3 displays the comparison between the output signals generated by the D-LPV model and the nonlinear model. The mean square error for $y_1(t)$, $y_2(t)$ and $y_3(t)$, between the nonlinear and the D-LPV model, are 1.08×10^{-5}, 3.45×10^{-4} and, 6.67×10^{-6}, respectively. It is clear, considering Fig. 3.3 and the small errors, that the D-LPV model matches the nonlinear model. To illustrate the performance under disturbance and noise, the following matrices are considered

$$
B_{d1} = B_{d2} = \dots = B_{d16} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, D_d = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.5 \end{bmatrix}
$$

Then, the following state-observer is designed

$$
\begin{aligned}
\dot{z}(t) &= \sum_{j=1}^{16} \rho_j(\hat{\zeta}(t))\left[N_j z(t) + G_j u(t) + L_j y(t)\right] \\
\hat{x}(t) &= z(t) + T_2 y(t)
\end{aligned}
\tag{3.71}
$$

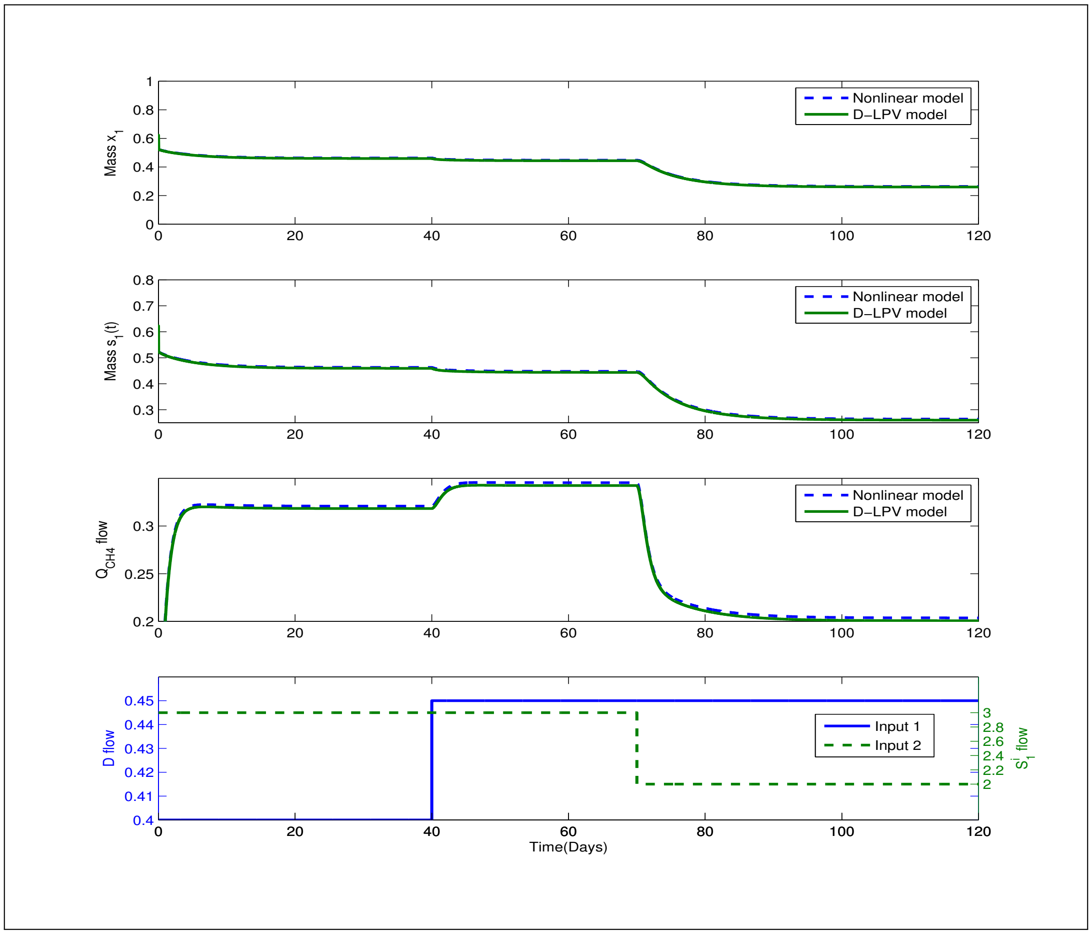


Figure 3.3 – Nonlinear system outputs and inputs.

Note the scheduling vector $\zeta(t)$ is considered unmeasurable and needs to be estimated. Therefore, the scheduling functions defined in (3.69) are expressed as $\rho_j(\hat{\zeta}(t))$, which indicates the estimation of the unmeasurable states $x_1(t)$, $x_2(t)$ and the output $u(t)$. The gain scheduling functions take the form

$$
\begin{aligned}
\hat{\zeta}_1 &= \hat{x}_1(t) \in [\bar{a}_1, \underline{a}_1] = [0.2\ 0.7], \\
\hat{\zeta}_2 &= \hat{x}_2(t) \in [\bar{a}_2, \underline{a}_2] = [0.01\ 0.6], \\
\hat{\zeta}_3 &= u_1(t) \in [\bar{a}_3, \underline{a}_3] = [0.2\ 0.8], \\
\hat{\zeta}_4 &= \frac{k_{m1}}{k_{s1} + \hat{x}_2(t)} I_{pH} \in [\bar{a}_4, \underline{a}_4] = [3.7\ 9.9].
\end{aligned}
$$

For each ζ_j, two local scheduling functions are constructed as

$$\mu_1^k(\hat{\zeta}_k) = \frac{\bar{a}_k - \hat{\zeta}_k}{\bar{a}_k - \underline{a}_k}, \mu_2^k = 1 - \mu_1^k, k = 1, 2...4. \tag{3.72}$$

The 16 scheduling functions are computed as

$$\rho_j(\hat{\zeta}(t)) = \prod_{k=1}^{4} \mu_{ik}(\hat{\zeta}_j) \tag{3.73}$$

The observer gain matrices are computed by solving Corollary 3.3.1 with the YALMIP toolbox (Lofberg, 2004). Matrices T_1, T_2 are computed independently from the main Corollary by solving (3.42), such that

$$T_1 = \mathsf{diag}([0.5, 0.5, 0.5, 0]), T_2 = \mathsf{diag}([0.5, 0.5, 0.5, 1]),$$

Matrices G_i can be easily computed by considering $G_j = T_1 B_i$, $\forall j = 1, 2, ..., 16$. Finally, by solving Corollary 3.3.1 (LMI (3.62)), the following matrices are computed

$$P_1 = P_1^T = \begin{bmatrix} 0.0008 & 0.0002 & 0.0006 & -0.0000 \\ * & 0.0002 & 0.0004 & -0.0000 \\ * & * & 0.0012 & -0.0000 \\ * & * & * & 0.9978 \end{bmatrix} > 0,$$

$$Q_1 = \begin{bmatrix} -0.0008742 & 0.0000181 & 0.0001883 & -0.0000013 \\ 0.0000181 & -0.0009441 & 0.0001036 & 0.0000024 \\ 0.0001883 & 0.0001036 & -0.0003484 & -0.0000047 \\ -0.0000013 & 0.0000024 & -0.0000047 & -0.0009465 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -0.0008742 & 0.0000181 & 0.0001883 & -0.0000013 \\ 0.0000181 & -0.0009441 & 0.0001036 & 0.0000024 \\ 0.0001883 & 0.0001036 & -0.0003484 & -0.0000047 \\ -0.0000013 & 0.0000024 & -0.0000047 & -0.0009465 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -0.0008731 & 0.0000191 & 0.0001899 & -0.0000013 \\ 0.0000191 & -0.0009434 & 0.000105 & 0.0000023 \\ 0.0001899 & 0.000105 & -0.0003459 & -0.0000047 \\ -0.0000013 & 0.0000023 & -0.0000047 & -0.0009465 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} -0.0008731 & 0.0000191 & 0.0001899 & -0.0000013 \\ 0.0000191 & -0.0009434 & 0.000105 & 0.0000023 \\ 0.0001899 & 0.000105 & -0.0003459 & -0.0000047 \\ -0.0000013 & 0.0000023 & -0.0000047 & -0.0009465 \end{bmatrix},$$

$$Q_5 = \begin{bmatrix} -0.0008687 & 0.0000209 & 0.0001951 & -0.0000013 \\ 0.0000209 & -0.0009429 & 0.0001066 & 0.0000023 \\ 0.0001951 & 0.0001066 & -0.0003409 & -0.0000046 \\ -0.0000013 & 0.0000023 & -0.0000046 & -0.0009465 \end{bmatrix},$$

$$Q_6 = \begin{bmatrix} -0.0008687 & 0.0000209 & 0.0001951 & -0.0000013 \\ 0.0000209 & -0.0009429 & 0.0001066 & 0.0000023 \\ 0.0001951 & 0.0001066 & -0.0003409 & -0.0000046 \\ -0.0000013 & 0.0000023 & -0.0000046 & -0.0009465 \end{bmatrix},$$

$$Q_7 = \begin{bmatrix} -0.0008682 & 0.0000214 & 0.0001959 & -0.0000013 \\ 0.0000214 & -0.0009425 & 0.0001074 & 0.0000023 \\ 0.0001959 & 0.0001074 & -0.0003395 & -0.0000046 \\ -0.0000013 & 0.0000023 & -0.0000046 & -0.0009465 \end{bmatrix},$$

$$Q_8 = \begin{bmatrix} -0.0008682 & 0.0000214 & 0.0001959 & -0.0000013 \\ 0.0000214 & -0.0009425 & 0.0001074 & 0.0000023 \\ 0.0001959 & 0.0001074 & -0.0003395 & -0.0000046 \\ -0.0000013 & 0.0000023 & -0.0000046 & -0.0009465 \end{bmatrix},$$

$$Q_9 = \begin{bmatrix} -0.0008668 & 0.0000207 & 0.0001962 & -0.0000047 \\ 0.0000207 & -0.0009453 & 0.0001057 & 0.0000088 \\ 0.0001962 & 0.0001057 & -0.0003394 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},$$

$$Q_{10} = \begin{bmatrix} -0.0008668 & 0.0000207 & 0.0001962 & -0.0000047 \\ 0.0000207 & -0.0009453 & 0.0001057 & 0.0000088 \\ 0.0001962 & 0.0001057 & -0.0003394 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},$$

$$Q_{11} = \begin{bmatrix} -0.0008662 & 0.0000215 & 0.000197 & -0.0000047 \\ 0.0000215 & -0.0009442 & 0.0001069 & 0.0000088 \\ 0.000197 & 0.0001069 & -0.0003383 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} -0.0008662 & 0.0000215 & 0.000197 & -0.0000047 \\ 0.0000215 & -0.0009442 & 0.0001069 & 0.0000088 \\ 0.000197 & 0.0001069 & -0.0003383 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},$$

$$Q_{13} = \begin{bmatrix} -0.000868 & 0.0000208 & 0.0001953 & -0.0000047 \\ 0.0000208 & -0.0009447 & 0.0001064 & 0.0000088 \\ 0.0001953 & 0.0001064 & -0.0003392 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},$$

$$Q_{14} = \begin{bmatrix} -0.000868 & 0.0000208 & 0.0001953 & -0.0000047 \\ 0.0000208 & -0.0009447 & 0.0001064 & 0.0000088 \\ 0.0001953 & 0.0001064 & -0.0003392 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},$$

$$
Q_{15} = \begin{bmatrix} -0.0008678 & 0.0000212 & 0.0001954 & -0.0000047 \\ 0.0000212 & -0.0009439 & 0.000107 & 0.0000088 \\ 0.0001954 & 0.000107 & -0.0003392 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},
$$

$$
Q_{16} = \begin{bmatrix} -0.0008678 & 0.0000212 & 0.0001954 & -0.0000047 \\ 0.0000212 & -0.0009439 & 0.000107 & 0.0000088 \\ 0.0001954 & 0.000107 & -0.0003392 & -0.0000123 \\ -0.0000047 & 0.0000088 & -0.0000123 & -0.0009459 \end{bmatrix},
$$

$$
P_2 = \begin{bmatrix} 0.0004 & 0.0000 & -0.0000 & 0.0000 \\ * & 0.0009 & 0.0002 & 0.0000 \\ * & * & 0.0009 & -0.0000 \\ * & * & * & 0.0019 \end{bmatrix} > 0,
$$

$$
X = \begin{bmatrix} -0.0018 & -0.2506 & 0.0334 & 0.0631 \end{bmatrix}, S = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T.
$$

The gain matrices can be computed by considering the matrices P, and Q_j according equations (3.48)-(3.49). The attenuation level obtained is $\gamma = 0.0611$. The small magnitude of the attenuation level guaranteed small amplification of the error due to disturbances and the unmeasurable gain scheduling functions.

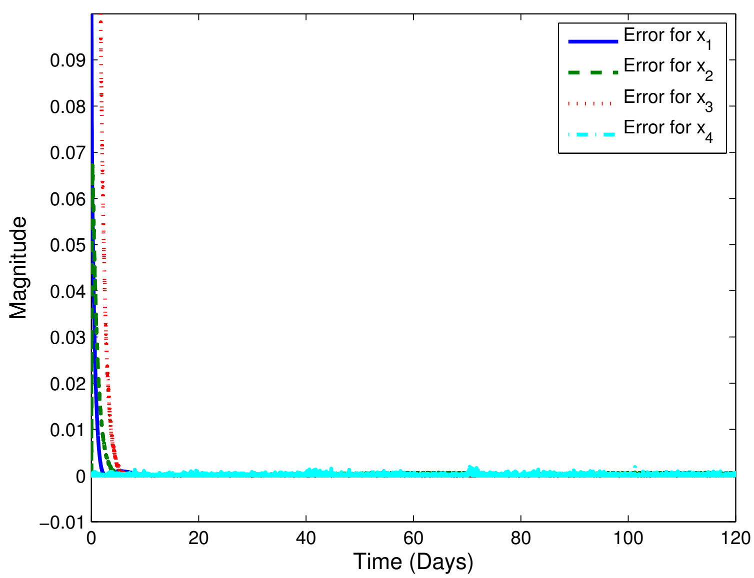


Figure 3.4 – Quadratic state estimation error

Fig. 3.4 shows the evolution of the estimation error. As can be observed, the estimation error converges asymptotic to zero even in the presence of disturbance and the error generated by the unmeasurable gain scheduling functions. The estimated scheduling functions are displayed in Fig. 3.5.

Summarizing, this approach can estimate the states and the scheduling functions with good performance and small error. Nevertheless, the performance can be optimized by considering a different mathematical transformation, which considers the convex property of the scheduling functions. This transformation will be detailed in the next section.

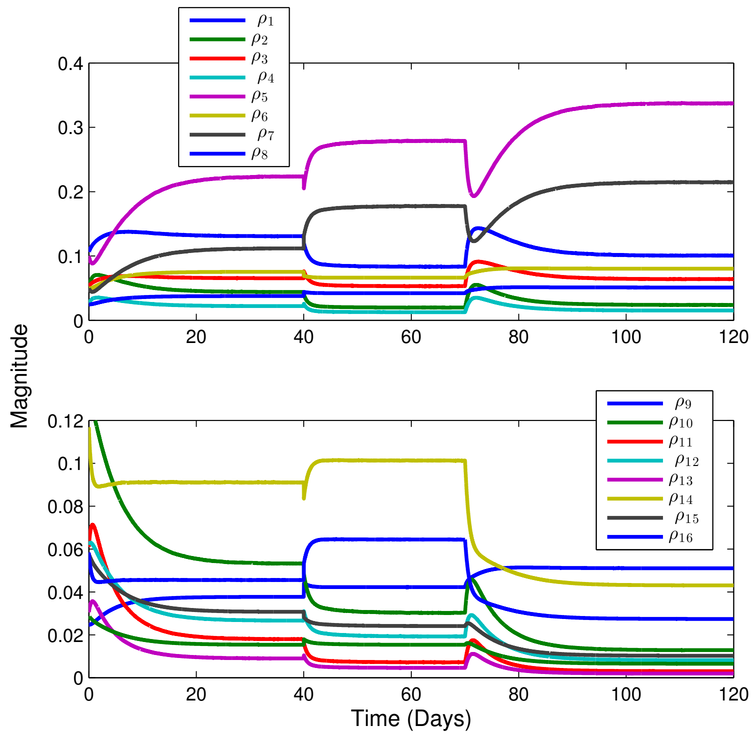


Figure 3.5 – Estimated gain scheduling functions

3.4 Approach 3: Observer design by considering the uncertain error system approach

This section addresses the problem of designing a suitable robust LPV observer for D-LPV systems with unmeasurable scheduling functions. To deal with the unmeasurable scheduling function, the error system with unmeasurable scheduling parameter is transformed into an uncertain error system with an estimated scheduling parameter. Similar to the previous section, H_∞ criterion is employed to guarantee flexibility and robustness performance, even in the presence of model-reality differences, disturbances, and the uncertainty provided by the unmeasurable scheduling parameter. Differently from Approach 2, which considers a unique solution with disturbance, the contribution presented in this section is separated, first by considering a system with disturbance, and then, a system with disturbance and noise. This separation is done because the results show, that by considering a disturbance vector the synthesized solution becomes more conservative in the second case rather than the first case.

3.4.1 Observer design without disturbances

Consider the D-LPV system as:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i(\zeta(t))\left[A_i x(t) + B_i u(t)\right] \\ y(t) &= Cx(t) \end{aligned} \tag{3.74}$$

Under convex property (3.8) and observability properties, the following standard LPV observer is considered

$$\dot{z}(t) = \sum_{j=1}^{h} \rho_j(\hat{\zeta}(t))\left[N_i z(t) + G_i u(t) + L_i y(t)\right]$$

$$\hat{x}(t) = z(t) + T_2 y(t) \tag{3.75}$$

$$r(t) = M(y(t) - C\hat{x}(t)) \tag{3.76}$$

Based on (3.74) and (3.76), the state-space error $e(t)$ is

$$e(t) = x(t) - \hat{x}(t) = (I - T_2C)x(t) - z(t)$$

assuming that there exists a $T_1 \in \mathbb{R}^{n\times n}$ matrix as $I - T_2C = T_1E$. The dynamic error is obtained as

$$\dot{e}(t) = \sum_{i=1}^{h} \rho_i T_1\left[A_i x(t) + B_i u(t)\right] - \sum_{i=1}^{h} \hat{\rho}_i\left[N_i T_1 E x(t) - N_i e(t) + G_i u(t) + L_i y(t)\right] \tag{3.77}$$

As pointed previously, the main challenge in the observer design, is to synthesize the error equation despite the differences between the scheduling functions ρ_i and $\hat{\rho}_i$. However, by considering the convex property of the scheduling function, the gain matrices of the observer can be rewritten as uncertain matrices, such that the term $-\sum_{i=1}^{h} \rho_i(\hat{x}(t))N_i T_1 E x(t)$ can be handled as

$$\left[\sum_{j=1}^{h}\left[\left(\rho_j - \hat{\rho}_j\right)\right] N_j T_1 E\right] x(t) - \sum_{i=1}^{h} \rho_i N_i T_1 E x(t), \tag{3.78}$$

the term $\sum_{i=1}^{h} \rho_i(\hat{x}(t))N_i e(t)$ can be handled as follows:

$$-\left[\sum_{j=1}^{h}\left[\left(\rho_j-\hat{\rho}_j\right)\right]N_j\right]e(t)+\sum_{i=1}^{h}\rho_i N_i e(t). \tag{3.79}$$

In the same way $-\sum_{i=1}^{h}\rho_i(\hat{x}(t))G_i u(t)$ can be handled as follows:

$$\left[\sum_{j=1}^{h}\left[\left(\rho_j-\hat{\rho}_j\right)\right]G_j\right]u(t)-\sum_{i=1}^{h}\rho_i G_i u(t), \tag{3.80}$$

and finally, the term $-\sum_{i=1}^{h}\rho_i(\hat{x}(t))L_i y(t)$ is rewritten as

$$\left[\sum_{j=1}^{h}\left[\left(\rho_j-\hat{\rho}_j\right)\right]L_j C\right]x(t)-\sum_{i=1}^{h}\rho_i L_i C x(t). \tag{3.81}$$

By putting together (3.79)-(3.81) with (3.78), the error equation is obtained as:

$$\begin{aligned}\dot{e}(t)=&\sum_{i=1}^{h}\rho_i(x(t))\{(T_1A_i-N_iT_1E-L_iC)x(t)+(T_1B_i-G_i)u(t)+N_ie(t)\\&+\sum_{j=1}^{h}\left[\left(\rho_j-\hat{\rho}_j\right)\right]\left((N_jT_1E-L_jC)x(t)\ +G_ju(t)-N_je(t)\right)\}\end{aligned} \tag{3.82}$$

In order to guarantee the convergence to zero of the error dynamics, the following constraints are considered

$$T_1A_i-L_iC-N_iT_1E=0, \tag{3.83}$$

$$G_i-T_1B_i=0. \tag{3.84}$$

By considering (3.83), the following equations are equivalent:

$$N_i=T_1A_i+K_iC \tag{3.85}$$

$$K_i=N_iT_2-L_i \tag{3.86}$$

A particular solution of matrices T_1 and T_2 is computed as

$$\begin{bmatrix}T_1 & T_2\end{bmatrix}=\begin{bmatrix}I_n & 0\end{bmatrix}\begin{bmatrix}E\\C\end{bmatrix}^{\dagger}. \tag{3.87}$$

The error equation becomes

$$\dot{e}(t) = \sum_{i=1}^{h} \rho_i \left\{ N_i e(t) + \sum_{j=1}^{h} \left[\left(\rho_j - \hat{\rho}_j\right) \right] \left[\left(T_1 A_j\right) x(t) + G_j u(t) - N_j e(t) \right] \right\}. \tag{3.88}$$

In order to construct a residual-state-space error system, the states are extended as $x_e(t) = \begin{bmatrix} e(t)^T & x(t)^T \end{bmatrix}^T$. As a result, by considering (3.88), (3.74) and (3.76), the residual state-space error system is rewritten as

$$\begin{aligned} \bar{E}\dot{x}_e(t) &= \sum_{i=1}^{h} \rho_i \left[\left(\bar{A}_i + \Delta\bar{A}\right) x_e(t) + \left(\bar{B}_i + \Delta\bar{B}\right) \bar{u}(t) \right] \\ r(t) &= \bar{C} x_e(t) \end{aligned} \tag{3.89}$$

with

$$\bar{E} = \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \bar{A}_i = \begin{bmatrix} N_i & 0 \\ 0 & A_i \end{bmatrix}, \bar{B}_i = \begin{bmatrix} 0 \\ B_i \end{bmatrix}, \bar{C} = \begin{bmatrix} MC & 0 \end{bmatrix}, \quad \bar{u}(t) = \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$$

and matrices $\Delta\bar{A}$ and $\Delta\bar{B}$ defined by

$$\Delta\bar{A} = H_A F_A \Phi_A, \Delta\bar{B} = H_B F_B \Phi_B,$$

with

$$H_A = \begin{bmatrix} \begin{bmatrix} T_1 A_1 & \dots & T_1 A_h \end{bmatrix} & \begin{bmatrix} N_1 & \dots & N_h \end{bmatrix} \\ 0 & 0 \end{bmatrix}, F_A = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix},$$

$$\Phi_A = \begin{bmatrix} I_A & 0 \\ 0 & -I_A \end{bmatrix}^T, I_A = \begin{bmatrix} I_{n_1} & \dots & I_{n_h} \end{bmatrix}^T,$$

$$H_B = \begin{bmatrix} \begin{bmatrix} G_1 & \dots & G_h \end{bmatrix} \\ 0 \end{bmatrix}, F_B = \begin{bmatrix} & F \end{bmatrix}, \Phi_B = I_B,$$

$$I_B = \begin{bmatrix} I_{m_1} & \dots & I_{m_h} \end{bmatrix}^T, F = \begin{bmatrix} (\rho_1 - \hat{\rho}_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (\rho_h - \hat{\rho}_h) \end{bmatrix}.$$

Note that this transformation is possible due to the convex property (3.8), which implies $F^T(t)F(t) \leq I$.

The problem is reduced to design the observer gains to reject the influence of the control input $u(t)$ and to maximize the robustness against the uncertainty provided by the unmeasurable scheduling function. Sufficient conditions to achieve this objective are given through the following theorem:

Theorem 3.4.1. Consider the system (3.74) and the observer (3.75). The estimation error (3.89) is asymptotically stable with attenuation level $\gamma > 0$, such that $\| \mathscr{G}_e \|_\infty < \gamma$, if there exist scalars $\varepsilon_A > 0$, $\varepsilon_B > 0$, matrices $Q = Q^T > 0$, Ξ_i, X, M, and $\mathscr{P} = PE + SX$, $P > 0$, such that there exists a solution to the following optimization problem $\forall i \in [1, 2, ..., h]$:

$$\begin{aligned} \min_{P,Q,X,M,\Xi_i,\varepsilon_A,\varepsilon_B} \quad & \gamma \\ \text{s.t.} \quad & \\ \Psi_i < \quad & 0 \end{aligned} \tag{3.90}$$

where Ψ_i is given in the next page. T_1 is given by

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}.$$

Then, the observer parameters are given by $K_i = Q^{-1}\Xi_i$ and the gain matrices defined in (3.83)-(3.86).

Proof. By considering the $\mathscr{L}_2$-gain from $u(t)$ to $r(t)$

$$J_{r_u} := \int_0^\infty r^T(t)r(t)dt - \gamma^2 \int_0^\infty u^T(t)u(t)dt < 0, \tag{3.91}$$

and following the lines of Theorem 3.2.1, it is straightforward to obtain the following LMI conditions

$$\bar{E}^T \mathscr{X} = X^T \mathscr{X} > 0 \tag{3.92}$$

$$\begin{bmatrix} \mathrm{He}\left((\bar{A}_i + \Delta\bar{A}_i)^T \mathscr{X} \right) & \mathscr{X}(\bar{B}_i + \Delta\bar{B}_i) & \bar{C}^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0. \tag{3.93}$$

$$
\Psi_i = \begin{bmatrix}
\mathrm{He}\left\{(T_1A_i)^T Q + C\Xi_i\right\} & 0 & 0 & [QT_1A_1 \dots QT_1A_h] & [QT_1A_1 + \Xi_1 C \dots QT_1A_h + \Xi_h C] & -\varepsilon_A I_A^T & 0 & [QG_1 \dots QG_h] & 0 & (MC)^T \\
* & \mathrm{He}\{A_i^T \mathscr{P}\} & \mathscr{P}^T B & 0 & 0 & 0 & \varepsilon_A I_A^T & 0 & 0 & 0 \\
* & * & -\gamma^2 I_m & 0 & 0 & 0 & 0 & 0 & \varepsilon_B \Phi_B^T & 0 \\
* & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 \\
* & * & * & * & * & * & * & * & * & -I_p
\end{bmatrix}.
$$

---

with $\mathscr{X} = \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}$. By considering the following

$$
\mathscr{M} = \begin{bmatrix} \mathrm{He}\left(\bar{A}_i^T \mathscr{X}\right) & \mathscr{X}\bar{B}_i & \bar{C}^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix}, \tag{3.94}
$$

the LMI (3.93) can be rewritten as

$$
\mathscr{M} + \begin{bmatrix} \mathrm{He}\left(\Delta\bar{A}_i^T \mathscr{X}\right) & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathscr{X}\Delta\bar{B}_i & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} < 0. \tag{3.95}
$$

LMI (3.95) is rewritten in an equivalent form as

$$
\mathscr{M} + \tilde{H}_A F_A \tilde{\Phi}_A + (\tilde{H}_A F_A \tilde{\Phi}_A)^T + \tilde{H}_B F_B \tilde{\Phi}_B + \left(\tilde{H}_B F_B \tilde{\Phi}_B\right)^T < 0
$$

with

$$\tilde{H}_A F_A \tilde{\Phi}_A = \begin{bmatrix} XH_A \\ 0 \\ 0 \end{bmatrix} F_A \begin{bmatrix} \Phi_A & 0 & 0 \end{bmatrix}$$

$$\tilde{H}_B F_B \tilde{\Phi}_B = \begin{bmatrix} XH_B \\ 0 \\ 0 \end{bmatrix} F_B \begin{bmatrix} 0 & \Phi_B & 0 \end{bmatrix}.$$

By considering Lemma 3.2.1, the previous inequality becomes

$$\mathscr{M} + \varepsilon_A \hat{\Phi}_A^T \hat{\Phi}_A + \varepsilon_B \hat{\Phi}_B^T \hat{\Phi}_B + \frac{1}{\varepsilon_A} \hat{H}_A^T \hat{H}_A + \frac{1}{\varepsilon_B} \hat{H}_B^T \hat{H}_B < 0. \tag{3.96}$$

By applying lemma 3.2.1 and the Schur complement, the inequality (3.96) becomes

$$\begin{bmatrix} \mathrm{He}\left(\bar{A}_i^T \mathscr{X}\right) & \mathscr{X}\bar{B}_i & \bar{C}^T & \mathscr{X}H_A & \varepsilon_A \Phi_A^T & X\mathscr{X}H & 0 \\ * & -\gamma^2 I & 0 & 0 & 0 & 0 & \varepsilon_B \Phi_B^T \\ * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_A I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_A I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_B I & 0 \\ * & * & * & * & * & * & -\varepsilon_B I \end{bmatrix} < 0. \tag{3.97}$$

In order to rearrange columns 3 and 7 from LMI (3.97), the LMI is post- and pre-multiplied by

$$\mathscr{I} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix},$$

and its transpose, respectively. The LMI (3.97) results in

$$
\begin{bmatrix}
\mathrm{He}\left(\bar{A}_i^T \mathscr{X}\right) & X\mathscr{X}B & XH_A & \varepsilon_A \Phi_A^T & XH_B & 0 & \bar{C}^T \\
* & -\gamma^2 I & 0 & 0 & 0 & \varepsilon_B \Phi_B^T & 0 \\
* & * & -\varepsilon_A I & 0 & 0 & 0 & 0 \\
* & * & * & -\varepsilon_A I & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_B I & 0 & 0 \\
* & * & * & * & * & -\varepsilon_B I & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} < 0. \tag{3.98}
$$

Note that by replacing $\mathscr{X} = \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}$, $Q = Q^T > 0$, and the extended matrices of (3.89), the following inequality is obtained

$$
\left[\begin{array}{cccccccccc}
\mathrm{He}\left\{(T_1 A_i)^T Q + C\Xi_i\right\} & 0 & 0 & [QT_1A_1 \dots QT_1A_h] & [QT_1A_1 + \Xi_1 C \dots QT_1A_h + \Xi_h C] & -\varepsilon_A I_A^T & 0 & [QG_1 \dots QG_h] & 0 & (MC)^T \\
* & \mathrm{He}\{A_i^T P\} & PB_i & 0 & 0 & 0 & \varepsilon_A I_A^T & 0 & 0 & 0 \\
* & * & -\gamma^2 I_m & 0 & 0 & 0 & 0 & 0 & \varepsilon_B \Phi_B^T & 0 \\
* & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 \\
* & * & * & * & * & * & * & * & * & -I_p
\end{array}\right] < 0. \tag{3.99}
$$

Finally, the equality constraint (3.92) can be eliminated by considering Lemma 3.2.2. As a result, the LMI (3.90) is obtained. This completes the proof. □

Theorem 3.4.1 guarantees a solution to the state estimation problem, despite the unmeasurable scheduling functions. However, the solution does not consider disturbances, which is considered below.

3.4.2 Observer design with disturbances

In the presence of noise, the system (3.74) can be represented as

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i(x(t))\left[A_i x(t) + B_i u(t) + B_d d(t)\right] \\ y(t) &= Cx(t) + D_d d(t) \end{aligned} \tag{3.100}$$

where D_d is a constant matrix of appropriate dimension. Then, by considering the system (3.100) and the observer (3.75), the estimation error becomes

$$\begin{aligned} &e(t) = x(t) - \hat{x}(t) \\ &e(t) = (I - T_2 C)x(t) - z(t) - T_2 D_d d(t), \end{aligned}$$

Under slow variation assumption such that $\dot{d}(t) \approx 0$, the dynamic error equation is given by

$$\dot{e}(t) = \sum_{i=1}^{h} \rho_i T_1\left[A_i x(t) + B_i u(t) + B_d d(t)\right] - \sum_{i=1}^{h} \hat{\rho}_i\left[N_i T_1 E x(t) - N_i e(t) + G_i u(t) + L_i C x(t) + L_i D_d d(t)\right]$$

The term $-L_i D_d d(t)$ is handled as

$$-\left[\sum_{j=1}^{h}\left[\left(\rho_j - \hat{\rho}_j\right)\right] L_j D_d\right] d(t) + \sum_{i=1}^{h} \rho_i L_j D_d d(t), \tag{3.101}$$

The remaining terms are handled as described in the previous section, such that the dynamical residual estimation error becomes

$$\dot{e}(t) = \sum_{i=1}^{h} \rho_i(x(t)) \{(T_1A_i - N_iT_1E - L_iC)x(t) + (T_1B_i - G_i)u(t) + N_ie(t) + (T_1B_d + K_jD_d)d(t)$$

$$+ \sum_{j=1}^{h} \left[\left(\rho_j - \hat{\rho}_j\right)\right] \left((N_jT_1E - L_jC)x(t) + G_ju(t) + N_je(t) - (N_jT_2D_d - L_jD_d)d(t)\right)\}$$

$$r(t) = Ce(t) + Dd(t) \quad . \tag{3.102}$$

Under gain synthesis (3.85)-(3.86), the estimation error becomes

$$\dot{e}(t) = \sum_{i=1}^{h} \rho_i \left[N_ie(t) + (T_1B_d + K_jD_d)d(t) + \sum_{j=1}^{h} \left[\left(\rho_j - \hat{\rho}_j\right)\right] \left[\left(T_1A_j\right)x(t) + G_ju(t) - N_je(t) - K_jD_dd(t)\right]\right].$$

Again, the states are augmented in order to obtain a compact representation, by considering $x_e(t) = \begin{bmatrix} e(t)^T & x(t)^T \end{bmatrix}^T$ such as

$$\bar{E}\dot{x}_e(t) = \sum_{i=1}^{h} \rho_i \left[\left(\bar{A}_i + \Delta\bar{A}_i\right)x_e(t) + (\bar{B}_i + \Delta\bar{B}_i)\bar{u}(t)\right] \tag{3.103}$$

$$r(t) = \bar{C}x_e(t)$$

with

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \bar{A}_i = \begin{bmatrix} N_i & 0 \\ 0 & A_i \end{bmatrix}, \qquad \bar{B}_i = \begin{bmatrix} 0 & (T_1B_d + K_jD_d) \\ B_i & B_d \end{bmatrix}, \bar{C} = \begin{bmatrix} MC & 0 \end{bmatrix},$$

$$\bar{D} = \quad [0\, MD_d], \quad \bar{u}(t) = \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}.$$

where $\Delta\bar{A}$ and $\Delta\bar{B}$ are defined by

$$\Delta\bar{A}_i = H_AF_A\Phi_A, \Delta\bar{B}_i = H_BF_B\Phi_B,$$

with

$$H_A = \begin{bmatrix} \begin{bmatrix} T_1A_1 & \dots & T_1A_h \end{bmatrix} & \begin{bmatrix} N_1 & \dots & N_h \end{bmatrix} \\ 0 & 0 \end{bmatrix}, F_A = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix},$$

$$\Phi_A = \begin{bmatrix} I_A & 0 \\ 0 & -I_A \end{bmatrix}^T, I_A = \begin{bmatrix} I_{n_1} & \dots & I_{n_h} \end{bmatrix}^T,$$

$$H_B = \begin{bmatrix} \begin{bmatrix} G_1 & \dots & G_h \end{bmatrix} & \begin{bmatrix} K_1D_d & \dots & K_hD_d \end{bmatrix} \\ 0 & 0 \end{bmatrix}, F_B = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \Phi_B = \begin{bmatrix} I_B & 0 \\ 0 & -I_B \end{bmatrix}$$

$$I_B = \begin{bmatrix} I_{m_1} & \dots & I_{m_h} \end{bmatrix}^T, F = \begin{bmatrix} (\rho_1 - \hat{\rho}_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (\rho_h - \hat{\rho}_h) \end{bmatrix}.$$

Finally, sufficient conditions to guarantee asymptotic convergence of the estimation error are given by the following theorem:

Theorem 3.4.2. There exists a robust state estimation observer (3.75) for the D-LPV system (3.100) with H_∞ attenuation level $\gamma > 0$, if there exist scalars $\varepsilon_A > 0$, $\varepsilon_B > 0$, matrices $Q = Q^T > 0$, Ξ_i, X, M, and $\mathscr{P} = PE + SX$, $P^T > 0$, $\forall i \in [1, 2, ..., h]$, such that there exists a solution to the following optimization problem:

$$\begin{aligned} \min_{P, Q, \Xi_i, \varepsilon_A, \varepsilon_B} \quad & \gamma \\ \text{s.t.} \quad & \\ \Psi_i < \quad & 0 \end{aligned} \tag{3.104}$$

where Ψ_i is given in the next page, and

$$\begin{aligned} \Psi_{15} &= [Q_1T_1B_d + \Xi_1D_d ... Q_hT_1B_d + \Xi_hD_d], \\ \Psi_{16} &= [QT_1A_1 + \Xi_1C \dots QT_1A_h + \Xi_hC], \\ \begin{bmatrix} T_1 & T_2 \end{bmatrix} &= \begin{bmatrix} I_n & 0 \end{bmatrix} \begin{bmatrix} E & B_d \\ C & 0 \end{bmatrix}^\dagger. \end{aligned}$$

The observer parameters are given by $K_i = Q^{-1}\Xi_i$ and the gain matrices defined in (3.83)-(3.86).

$$
\Psi_i = \left[\begin{array}{ccccccccccccc}
\mathrm{He}\left\{(T_1A_i)^T Q + C\Xi_i\right\} & 0 & 0 & \Psi_{15} & [QT_1A_1 \dots QT_1A_h] & \Psi_{16} & \varepsilon_A I_A^T & 0 & [QG_1 \dots QG_h] & [\Xi_1 D_d \dots \Xi_{ih} D_d] & 0 & 0 & (MC)^T \\
* & \mathrm{He}\{A_i^T \mathscr{P}\} & \mathscr{P}B_i & P\mathscr{P}B & 0 & 0 & 0 & -\varepsilon_A I_A^T & 0 & 0 & 0 & 0 & 0 \\
* & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_B \Phi_B^T & & 0 \\
* & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_B \Phi_B^T & (MD)^T \\
* & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon_A I_{n\times h} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\varepsilon_B I_{q\times h} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\varepsilon_B I_{m\times h} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -I_p
\end{array}\right]
$$

Proof. The proof follows the line of proof of Theorem 3.4.1, therefore it is omitted. □

The main advantage of the approach is to consider the convex property of the polytopic system, which guarantees good robustness conditions. Nevertheless, from the numerical point of view, finding an optimal and feasible solution for the LMIs could be a difficult task due to the size of the LMI. From a practical point of view, a useful solution, even for non-modeled noise can be obtained by solving Theorem 3.4.1 instead of Theorem 3.4.2, as considered in the early work (López-Estrada et al., 2014c).

3.5 Comparison between the three approaches

This section is dedicated to simulate and analyze a common numerical example to compare the performance of the proposed methods. The system in consideration is the system from the example (3.2.2) with additional disturbance matrices

$$B_d = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}, D_d = \begin{bmatrix} 0.5 \\ 0.3 \\ 0 \end{bmatrix}.$$

For each one of the approaches, the observer gain matrices have been computed with the Yalmip Toolbox (Lofberg, 2004). The observer gains for the first observer are the same than the states matrices given in the Example 3.2.2.

The observer gains for the ***second observer*** (3.34)-(3.35) are synthesized as follows: Matrices T_1 are computed from (3.42) as

$$T_1 = \begin{bmatrix} 0.9032 & 0.1613 & 0 \\ 0.1613 & 0.7312 & 0.1613 \\ 0.1613 & -0.2688 & 0.0000 \end{bmatrix}, T_2 = \begin{bmatrix} 0.0968 & -0.1613 & 0.1613 \\ -0.1613 & 0.2688 & -0.2688 \\ -0.1613 & 0.2688 & 0.7312 \end{bmatrix},$$

matrices G_j can be computed easily from (3.47). As as result of the solution of Theorem (Theorem 3.3.1, LMI 3.62), the following matrices are computed

$$\begin{aligned}
P_1 &= \begin{bmatrix} 7.0134 & -10.5890 & -22.6454 \\ * & 19.6435 & 49.8004 \\ * & * & 140.9729 \end{bmatrix}, Q_1 = \begin{bmatrix} -114.7529 & 182.9066 & -133.1926 \\ 182.9066 & -315.6476 & 175.7288 \\ -133.1926 & 175.7288 & -541.2855 \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} -194.6092 & 314.7322 & 0.8101 \\ 314.7322 & -537.1654 & -60.1559 \\ 0.8101 & -60.1559 & -524.2922 \end{bmatrix}, Q_3 = \begin{bmatrix} -28.9174 & 39.6077 & -36.7537 \\ 39.6077 & -76.6853 & 14.0845 \\ -36.7537 & 14.0845 & -375.9925 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 0.3495 & -0.0714 & 0.0000 \\ * & 0.2342 & -0.0000 \\ * & * & 125.8689 \end{bmatrix}, S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, X = \begin{bmatrix} -0.1929 & -0.0858 & -0.1154 \end{bmatrix}
\end{aligned}$$

.

Finally the gain matrices are computed by considering the mathematical relations (3.48)-(3.49), as follows:

$$N_1 = \begin{bmatrix} -7.2646 & -3.0454 & 10.6477 \\ 4.4532 & -17.7157 & 24.289 \\ -2.3009 & 8.8091 & -12.3629 \end{bmatrix}, N_2 = \begin{bmatrix} -8.1266 & 0.4551 & 1.7131 \\ 1.5608 & -6.0598 & 0.5132 \\ -0.8554 & 2.9643 & -0.4338 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} -6.7604 & 0.3116 & 1.6776 \\ 1.3608 & -5.9825 & 0.3096 \\ -0.7565 & 2.9252 & -0.332 \end{bmatrix}, L_1 = \begin{bmatrix} -1.3677 & 8.0795 & -4.8295 \\ -0.8933 & 16.2401 & -17.0734 \\ 0.4838 & -8.1604 & 8.577 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -0.9367 & 2.7676 & 0.4824 \\ 2.553 & 1.5442 & -2.3776 \\ -1.239 & -0.7875 & 1.2041 \end{bmatrix}, L_3 = \begin{bmatrix} -0.6198 & 2.8515 & 0.3985 \\ 2.6529 & 1.4249 & -2.2582 \\ -1.2884 & -0.7274 & 1.1441 \end{bmatrix}.$$

The observer gains for the ***third observer*** (3.75)-(3.76) are computed as follows: Matrices T_1, T_2 are computed independently from the main Corollary by solving (3.87), such that

$$T_1 = \begin{bmatrix} 0.5000 & 0 & 0 \\ 0 & 0.6667 & 0 \\ 0 & -0.3333 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0.5000 & 0 & 0 \\ 0 & 0.3333 & -0.3333 \\ 0 & 0.3333 & 0.6667 \end{bmatrix},$$

Matrices G_i can be easily computed by $G_j = T_1 B_i$, $\forall j = 1,2,...,16$. And from Theorem 3.4.2 (LMI (3.104)), we compute the following matrices:

$$Q = \begin{bmatrix} -0.0003 & -0.0001 & 0.0002 \\ * & 39.9826 & 79.9656 \\ * & * & 159.9305 \end{bmatrix} > 0, \Xi_i = 1 \times 10^{-4} \begin{bmatrix} -0.5240 & -0.2961 & 0.3277 \\ -0.2313 & -0.1865 & 0.0047 \\ -0.0999 & -0.1765 & -0.1696 \end{bmatrix},$$

$$\Xi_2 = 1 \times 10^{-4} \begin{bmatrix} -0.5114 & -0.2910 & 0.3203 \\ -0.2029 & -0.1754 & -0.0069 \\ -0.1354 & -0.1905 & -0.1541 \end{bmatrix}, \Xi_3 = 1 \times 10^{-4} \begin{bmatrix} -0.6684 & -0.1863 & 0.0261 \\ -0.4039 & -0.8608 & 0.4363 \\ -0.5455 & -0.5214 & -0.6473 \end{bmatrix},$$

$$P = P^T = \begin{bmatrix} 0.0014 & 0.0002 & 0 \\ * & 0.0001 & 0 \\ * & * & 1.9999 \end{bmatrix}, X = \begin{bmatrix} -0.0015 & -0.0001 & -0.0002 \end{bmatrix}$$

And the gain matrices

$$N_1 = \begin{bmatrix} -4.4679 & 2.8058 & 3.1944 \\ 0.9418 & -3.8936 & -0.7838 \\ -0.4709 & 1.9468 & 0.3919 \end{bmatrix}, N_2 = \begin{bmatrix} -4.4303 & 2.8202 & 3.1958 \\ 2.8959 & -2.9112 & -0.7846 \\ -1.4479 & 1.4556 & 0.3923 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} -3.2005 & 2.0432 & 3.6723 \\ 2.7012 & -2.0956 & -0.7838 \\ -0.4709 & 1.9468 & 0.3919 \end{bmatrix}, L_1 = \begin{bmatrix} -2.7660 & 1.6942 & 1.5558 \\ 0.8624 & -1.3322 & 0.4989 \\ -0.4312 & 0.6661 & -0.2494 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -2.7848 & 1.6852 & 1.5648 \\ 1.8854 & -0.9874 & 0.1541 \\ -0.9727 & 0.4937 & -0.0770 \end{bmatrix}, L_3 = \begin{bmatrix} -2.3998 & 2.3620 & 0.888 \\ 1.9828 & -1.6824 & 0.8491 \\ -0.9914 & 0.8412 & -0.4245 \end{bmatrix}.$$

The computed attenuation levels obtained by solving the proposed Theorems are shown in Table 3.2. Smaller values of the attenuation levels guarantee the objective performance and the robustness against the uncertainty provided by the scheduling functions for all the approaches. By taking in consideration the previous remark, it is clear that the best attenuation level is given by the method proposed with the Approach 1. Nevertheless, the other methods also give a small attenuation level.

Table 3.2 – Performance comparison

Observer \ Performance	γ
Approach 1	0.312
Approach 2	0.5832
Approach 3	1.76

Simulations are done by considering the same initial condition $x(0) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$, $\hat{x}(0) = \begin{bmatrix} 0.1 & -2 & -3 \end{bmatrix}^T$. The input is a sinusoidal signal, as displayed in Fig 3.7b. The included disturbance $d(t)$ is zero-mean noise with standard deviation 0.3. Note that the observer synthesized with the Approach 1 is also simulated with disturbances. The estimation errors $e_{xi}(t)$ are shown in Fig. 3.6a-c. All the observers converge with an error close to zero. The disturbance signal is well attenuated due to the H_∞ attenuation level γ. For example, considering the Approach 2, if the input signal is bounded by 4 and the disturbance is bounded by 0.3 then the error is bounded by 0.69984, which is small considering the magnitude of the system response. Of course, from Fig. 3.6, it is clear to see that the error levels are inferior to the bounds for all approaches. It must be noted that approach 1 was not designed to reject disturbances. This is the reason why the level of error is slightly bigger than the other two approaches, but still small. Nevertheless, even when the Approach 1 presents the biggest estimation error, it also presents the faster asymptotic convergence time. For example, for the Approaches 2 and 3, the observer converges in 2 seconds while

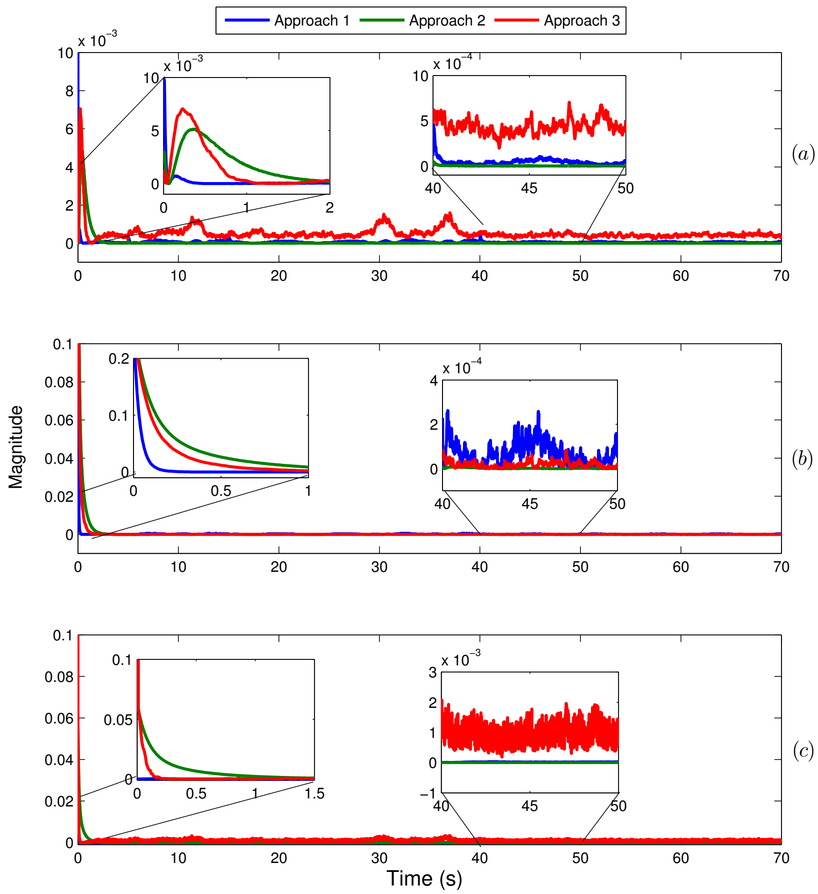


Figure 3.6 – State estimation errors (Fault free case) e_x. a) e_{x_1}, b) e_{x_2}, c) e_{x_3}.

the observer designed by the method proposed in the Approach 1 converges in 1 second.

The estimated scheduling functions are displayed in Fig. 3.7a. Only the estimation given by the Approach 2 is presented because a good performance on the state estimation indicates also good performance on the scheduling function estimation. In other words, all the estimated gain functions are similar. As can be observed from Fig. 3.7a., the scheduling functions represent the interactions between the 3 models during the simulation time. The scheduling functions also show how the operation region is periodically changing due to the nature of the input signal.

A second analysis can be done by considering singular plots of the frequency response of the error systems (3.13), (3.51), and (3.103), corresponding to the first, second and the third observer, respectively. Nevertheless due that the systems are time-varying, frequency responses for the all convex combinations cannot be generated. Instead, it is necessary to consider the scheduling functions values for a specific t,

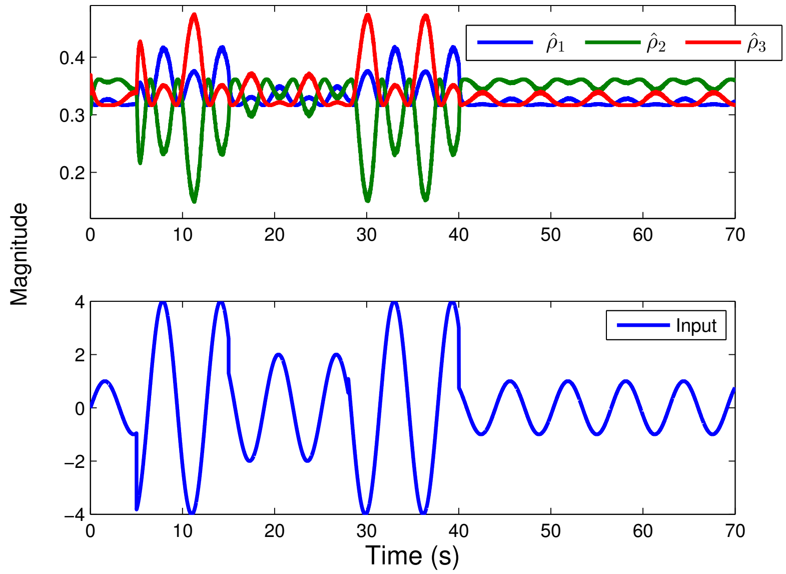


Figure 3.7 – Estimated gain scheduling functions and the applied input.

and then compute the parameter-varying matrices[2]. For instance, the transfer function of the error system (3.51) can be expressed as

$$G_{r\bar{d}}^{\hat{\rho}}(s) = \frac{r(s)}{\bar{d}(s)} = \bar{C}\left(s\bar{E} - \bar{A}^{\hat{\rho}}\right)^{-1} B^{\hat{\rho}}\bar{d}(s) + \bar{D}(s) \tag{3.100}$$

where $\bar{E}, \bar{A}^{\hat{\rho}}, \bar{B}^{\hat{\rho}}, \bar{C}$, and $\bar{D}$ are the states matrices of (3.51) computed at the instant t for given values of the estimated scheduling functions $\hat{\rho}$. For example at $t = 10$s, the gain scheduling values are $\hat{\rho}_1 = 0.4048$, $\hat{\rho}_2 = 0.5258$, and $\hat{\rho}_3 = 0.0694$. The parameter varying matrices of (3.51) are computed as

[2] As remarked, the analysis is done just for illustrative purposes. Different analysis can be done by considering the attenuation of the disturbances as detailed in Chapter 4. Also, another analysis could be done by considering just vertex of the D-LPV system.

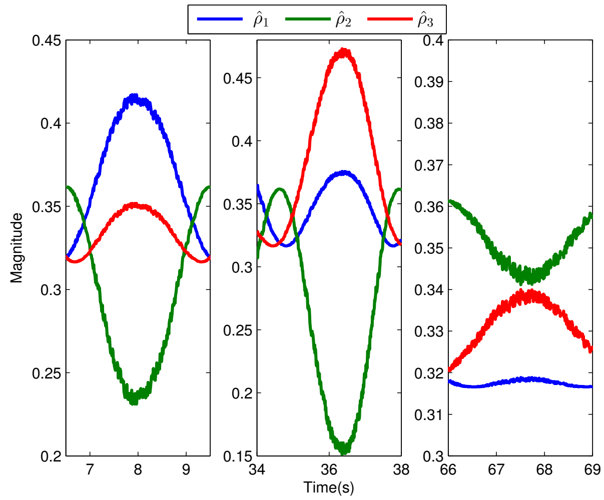
$$\bar{A} = \begin{bmatrix} -46.9092 & -16.5403 & 0.2116 & 0 & 0 & 0 \\ -83.4148 & -0.974424 & -200.9839 & -0.8047 & -0-4023 & 0 \\ 0.2155 & -0.0021 & -0.5568 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9.4380 & 5 & 6.5 \\ 0 & 0 & 0 & 2.9808 & -5 & -1.25 \\ 0 & 0 & 0 & -7.5544 & 4 & 7.7286 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 & -9.1793 \\ 0.0382 & -46.4176 \\ 0 & -0.0010 \\ 0 & 0.7522 \\ 0.8457 & 0 \\ 0.5511 & 0.5000 \end{bmatrix}.$$

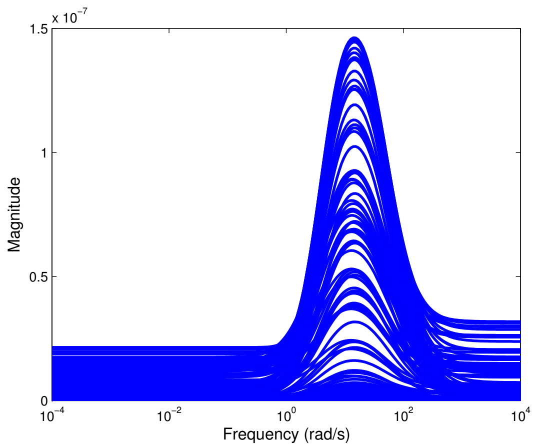
Then, a transfer function can be synthesized as given in (3.100). Finally, a singular value plot can be generated with the control system toolbox of MATLAB the "*sigma*plot" function. Note that similar analysis can be done for the error systems (3.13) and (3.103). Nevertheless one point has not enough information about the global dynamic. Then, for illustrative purposes, representative combinations of the scheduling functions are selected from Fig. 3.7a. Fig. 3.8a shows the selected combinations of the scheduling functions during selected interval-times. Then for each point in the intervals, a singular value plot is generated[3].

Fig. 3.8b displays the singular value plot for the first approach. This approach perform the smallest singular values. Nevertheless, these plots are not representatives because this approach does not considers disturbances attenuations and therefore, it is logical to obtain singular values close to zero that represent only the robustness against the unmeasurable scheduling functions. Fig. 3.8c shows the singular values plots of the second approach. The results show that this observer presents better performance at small frequency disturbances, while the bigger amplification of the disturbances are presented at high frequencies. Even that, the biggest disturbance amplification is 0.07 which is small and guarantee good robustness and state estimation. Fig. 3.8d shows the singular values of the third approach. In this case, the worst performance is presented at frequencies close to 10^0 rad/s.

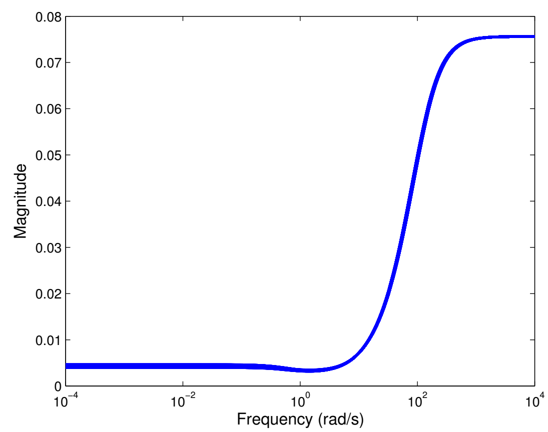
[3] The intervals are selected heuristically by visual analysis of the estimated gain scheduling functions displayed in Fig. 3.7a. For each point in the interval-time a singular plot is generated. As a result, each plot contains the results of 200 convex combinations.



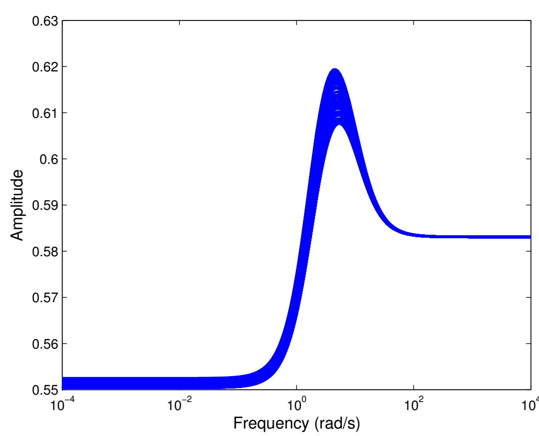
(a) Selected intervals of the estimated gain scheduling functions.



(b) Singular plots of the Approach 1



(c) Singular plots of the Approach 2



(d) Singular plots of the Approach 3

Figure 3.8 – Singular value plots.

In conclusion, the results showed in the common example illustrate the effectiveness of the proposed methods to perform state estimation, and attenuate disturbances, despite the error generated by the unmeasurable gain scheduling functions. Moreover, by comparing the results of the three approaches, is clear that for the proposed example the best trade-off between robustness and state estimation is given by the second approach.

The next step is to perform fault detection and isolation based on the proposed methods, as it will be analyzed in the following section.

3.6 Application to robust fault detection and fault isolation

In previous sections, different methods were derived for designing robust state observers for D-LPV systems with unmeasurable scheduling functions. In addition to the previous sections, the purpose of this part is to use previous results to generate residuals to detect and isolate sensor faults. In order to guarantee robustness in the generated residuals, the observer must be only sensitive to faults, even in the presence of disturbance and model-reality differences, as pointed in (Chen and Patton, 1999), such that during fault-free operation, the magnitude of the residuals should be zero, or close to. In the presence of a fault, the residual should change its value, bigger than a predefined threshold, as an indication of fault occurrence. The residual generation is a process for extracting fault symptoms from the system, with the fault symptom represented by the residual signal (Chen and Patton, 1999). This step is necessary to avoid critical consequences and helps in taking appropriate decisions, either by shutting down the system safely, or continuing the operation in degraded mode, despite the presence of the fault. To reach this objective, a generalized observer scheme (GOS), as proposed in (Frank, 1990; Isermann, 2006), can be considered. This process is described as follows.

3.6.1 Sensor fault detection and isolation

Firs, it is considered a D-LPV system under sensor faults as

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i \left[A_i x(t) + B_i u(t) + B_{di} d(t)\right] \\ y(t) &= Cx(t) + D_d d(t) + D_f f(t), \end{aligned} \tag{3.101}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $d(t)$, and $f(t)$ are the state vector, the control input, the disturbance input, the measured output vector, and the sensor fault vector, respectively. E, A_i, B_i, B_{di}, C and D_f are constant matrices of appropriate dimensions.

Then, by considering the GOS scheme, p observers are synthesized, where p is the number of sensor faults under consideration. A subsystem insensitive to a component f_p of the fault vector $f(t)$ is extracted for each observer by deriving the output vector $y(t)$. In order to isolate the sensor faults, a normalized residual vector is generated such that its p^{th} component is sensitive to all faults but p^{th} one. The bank of p fault diagnosis observers is given by[4]:

$$\begin{aligned} \dot{z}^p(t) &= \sum_{j=1}^{h} \hat{\rho}_j \left[N_j^p z(t) + G_j^p u(t) + L_j^p C^p x(t) \right] \\ \hat{x}^p(t) &= z^p(t) + T_2 C^p x(t). \end{aligned} \tag{3.102}$$

which generates p normalized residual expressed as

$$r^p(t) = \| M^p(y^p(t) - C^p \hat{x}^p(t)) \| \tag{3.103}$$

where $z^p(t)$ represent the estimated vector, $\hat{x}^p(t)$ the estimated states, of the p^{th} observer. N_j^p, G_j^p, L_j^p, T_2^p for p unknown matrices of appropriate dimension. Then, the observer gain matrices are computed for each set of given input matrices C^p, such that the robustness and convergence are validated for the p^{th} observer.

At this stage, the procedure is to compare the normalized residuals r to some predefined thresholds according to a test and at a stage where symptoms $S(r)$ are produced. To make possible to detect a fault, a set of structured residuals is required. The bank of observers generates an incidence matrix as shown in Table 3.3. Each column is called the coherence vector associated to each fault signature.

Table 3.3 – Incidence matrix

Symptom / Residual	S_0	S_1	S_2	$\dots$	S_p
$\| r^1 \|$	0	0	1	1	1
$\| r^2 \|$	0	1	1	1	1
$\dots$	0	1	1	0	1
$\| r^p \|$	0	1	1	1	0

It is clear from Table 3.3 that the decoupled observer method provides an efficient FDI technique for sensor faults. In the presence of a sensor fault, the observer, insensitive to the fault, estimates state vector $\hat{x}(t)$ and consequently estimates the output corrupted by the fault. Decision making can be carried-out according to an elementary binary logic.

[4]For illustrative purpose, only the standard observer is presented here. However the same logic is applicable for each approach.

3.6.2 Comparison between the three approaches under faults

To illustrate the performance of each method under faults, simulations are done by considering the example given in Sec. 3.5. Same simulation conditions are considered for each method, noise, initial conditions, as given previously. To provide useful residuals for the fault diagnosis problem, a generalized bank of observer in the GOS formulation is built for each approach. Observability assumptions are verified for each one of the observers. The synthesis of each observer is computed with the Yalmip toolbox (Lofberg, 2004). The computed matrices for the **first observer bank (Approach 1)** are

Table 3.4 – Gain matrices for the first observer bank

	Observer 1	Observer 2	Observer 3
P^p	$\begin{bmatrix} 0.0003 & 0.0001 & 0 \\ 0.0001 & 0.008 & 0 \\ 0 & 0 & 64.6776 \end{bmatrix}$	$\begin{bmatrix} 0.0134 & 0.0004 & 0 \\ 0.0004 & 0.0006 & 0 \\ 0 & 0 & 95.6541 \end{bmatrix}$	$\begin{bmatrix} 0.0119 & 0.0001 & 0 \\ 0.0001 & 0.0002 & 0 \\ 0 & 0 & 53.1749 \end{bmatrix}$
Ξ_i^p	$\begin{bmatrix} \begin{bmatrix} -0.0002 & 0.0002 \\ 0.1673 & -0.3289 \\ 0.1617 & 0 \end{bmatrix} & \begin{bmatrix} 0.0002 & -0.0002 \\ 0.1674 & -0.329 \\ 0.1617 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0.0259 & -0.0509 \\ 0.025 & 0 \end{bmatrix} & \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} 0.0248 & 0 \\ 0 & 0 \\ 0 & 0.0239 \end{bmatrix} & \begin{bmatrix} 0.0247 & 0 \\ 0 & 0 \\ 0 & 0.0239 \end{bmatrix} \\ \begin{bmatrix} 0.0241 & 0 \\ 0 & 0 \\ 0 & 0.0229 \end{bmatrix} & \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} 0.0137 & 0 \\ 0 & 0.0067 \\ 0 & 0.0067 \end{bmatrix} & \begin{bmatrix} 0.0187 & 0 \\ 0 & 0.0067 \\ 0 & 0.0037 \end{bmatrix} \\ \begin{bmatrix} 0.0137 & 0 \\ 0 & 0.0067 \\ 0 & 0.0067 \end{bmatrix} & \end{bmatrix}$
X_1^p	$\begin{bmatrix} -0.0000002 & 0.0000036 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0000137 & -0.0000004 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0000123 & -4.000 \times 10^{-08} & -1.000 \times 10^{-08} \end{bmatrix}$
Q^p	$\begin{bmatrix} 0.141 & -0.0192 & 0 \\ -0.0192 & 0.0572 & 0 \\ 0 & 0 & 0.6468 \end{bmatrix}$	$\begin{bmatrix} 0.0117 & -0.0034 & 0 \\ -0.0034 & 0.007 & 0 \\ 0 & 0 & 0.0957 \end{bmatrix}$	$\begin{bmatrix} 0.0217 & 0.0046 & 0 \\ 0.0046 & 0.0044 & 0 \\ 0 & 0 & 0.0532 \end{bmatrix}$
X_2^p	$\begin{bmatrix} 0.0111 & -0.0141 & -0.028 \end{bmatrix}$	$\begin{bmatrix} 0.0151 & -0.0188 & -0.0223 \end{bmatrix}$	$\begin{bmatrix} -0.0217 & -0.0047 & -0.0046 \end{bmatrix}$

Matrices for the **second observer bank (Approach 2)**

Table 3.5 – Matrices for the second observer bank

	Observer 1	Observer 2	Observer 3
P^p	$\begin{bmatrix} 0.2444 & -0.0483 & -0.0024 \\ -0.0483 & 0.0822 & 0.0001 \\ -0.0024 & 0.0001 & 6.4585 \end{bmatrix}$	$\begin{bmatrix} 0.83 & 0.0483 & -0.023 \\ 0.0483 & 0.6145 & -0.005 \\ -0.023 & -0.005 & 15.2401 \end{bmatrix}$	$\begin{bmatrix} 0.253 & -0.0148 & 0.0071 \\ -0.0148 & 2.1675 & 2.1604 \\ 0.0071 & 2.1604 & 2.1668 \end{bmatrix}$
Q_j^p	$\begin{bmatrix} \begin{bmatrix} -1.6606 & 0.0037 \\ -5.2447 & -6.6313 \\ -3.9806 & -15.9277 \end{bmatrix} & \begin{bmatrix} -1.9161 & 0.2473 \\ -5.2813 & -6.5929 \\ -3.9837 & -15.9256 \end{bmatrix} \\ \begin{bmatrix} -4.6318 & 3.0126 \\ -4.2629 & -7.62 \\ -4.024 & -15.8844 \end{bmatrix} & \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} -4.843 & -12.5566 \\ -1.8254 & 0.5259 \\ -2.291 & -13.0202 \end{bmatrix} & \begin{bmatrix} -4.7177 & -12.556 \\ -0.29 & 0.4952 \\ -2.3043 & -13.0213 \end{bmatrix} \\ \begin{bmatrix} -3.8231 & -12.5498 \\ -0.321 & 0.4887 \\ -2.3317 & -13.0212 \end{bmatrix} & \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} -3.1014 & -7.741 \\ -2.0188 & -5.2137 \\ -2.0802 & -5.3822 \end{bmatrix} & \begin{bmatrix} -3.1684 & -7.7143 \\ -1.9973 & -5.2213 \\ -2.0997 & -5.3765 \end{bmatrix} \\ \begin{bmatrix} -2.8296 & -7.7319 \\ -2.0171 & -5.2203 \\ -2.0904 & -5.3767 \end{bmatrix} & \end{bmatrix}$
P_2^p	$\begin{bmatrix} 0.83 & 0.0483 & -0.023 \\ 0.0483 & 0.6145 & -0.005 \\ -0.023 & -0.005 & 15.2401 \end{bmatrix}$	$\begin{bmatrix} 2.7314 & 1.1096 & 0 \\ 1.1096 & 1.1528 & 0 \\ 0 & 0 & 15.4392 \end{bmatrix}$	$\begin{bmatrix} 0.326 & -0.0954 & 0 \\ -0.0954 & 0.2722 & 0 \\ 0 & 0 & 4.5273 \end{bmatrix}$
X^p	$\begin{bmatrix} 0.0873 & -0.3334 & -0.1982 \end{bmatrix}$	$\begin{bmatrix} 0.3206 & -2.5293 & -3.0182 \end{bmatrix}$	$\begin{bmatrix} -0.0897 & -0.1748 & -0.1578 \end{bmatrix}$

The gain matrices can be computed by considering the matrices P, and Q_j according equations (3.48)-(3.49). Finally, the gain matrices for the **third observer bank (Approach 3)** are computed as

Table 3.6 – Matrices for the third observer bank

	Observer 1	Observer 2	Observer 3
Q^p	$\begin{bmatrix} 0.0006 & -0.0004 & 0.0009 \\ -0.0004 & 31.9809 & 63.969 \\ 0.0009 & 63.969 & 127.9385 \end{bmatrix}$	$\begin{bmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0.3 \\ 0.2 & 0.3 & 1996.5 \end{bmatrix}$	$\begin{bmatrix} -0.0016 & 0.0001 & -0.0004 \\ 0.0001 & 79.9614 & 79.9605 \\ -0.0004 & 79.9605 & 79.9611 \end{bmatrix}$
Ξ_i^p	$\begin{bmatrix} \begin{bmatrix} -0.00022 & 0.1747 \\ -0.2854 & 0.0841 \\ -0.2019 & -0.0646 \end{bmatrix} & \begin{bmatrix} -0.00021 & 0.00016 \\ -0.0002537 & 0.00005 \\ -0.0001981 & -0.00006 \end{bmatrix} \\ \begin{bmatrix} -0.0001341 & -0.0000058 \\ -0.0007477 & 0.0003879 \\ -0.0004461 & -0.0003362 \end{bmatrix} & \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} -0.003 & 0 \\ -0.0003 & -0.0003 \\ -0.0006 & -0.0018 \end{bmatrix} & \begin{bmatrix} -0.0031 & 0 \\ -0.0006 & -0.0003 \\ -0.0013 & -0.0018 \end{bmatrix} \\ \begin{bmatrix} -0.0043 & -0.0011 \\ -0.0014 & -0.001 \\ -0.0038 & -0.006 \end{bmatrix} & \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} -0.00047 & -0.00002 \\ -0.00017 & -0.00014 \\ -0.00014 & -0.00017 \end{bmatrix} & \begin{bmatrix} -0.00047 & -0.00002 \\ -0.0001 & -0.00015 \\ -0.00013 & -0.00016 \end{bmatrix} \\ \begin{bmatrix} -0.00071 & -0.00021 \\ -0.00044 & -0.00053 \\ -0.00040 & -0.00057 \end{bmatrix} & \end{bmatrix}$
P_2^p	$\begin{bmatrix} 0.0075 & 0.0008 & 0 \\ 0.0008 & 0.0005 & 0 \\ 0 & 0 & 1.5995 \end{bmatrix}$	$\begin{bmatrix} 0.0453 & 0.0049 & 0 \\ 0.0049 & 0.0028 & 0 \\ 0 & 0 & 1.9957 \end{bmatrix}$	$\begin{bmatrix} 0.0061 & 0.0007 & 0 \\ 0.0007 & 0.0003 & 0 \\ 0 & 0 & 1.5994 \end{bmatrix}$
X^p	$\begin{bmatrix} -0.0082 & -0.0007 & -0.0013 \end{bmatrix}$	$\begin{bmatrix} -0.0492 & -0.0046 & -0.0082 \end{bmatrix}$	$\begin{bmatrix} -0.0066 & -0.0007 & -0.0007 \end{bmatrix}$

The gain matrices can be computed by considering the matrices Q, and Ξ_i according equations (3.85)-(3.86).

Table 3.7 displays the computed attenuation levels for each one of the observers. An analysis of the disturbance performance can be made as discussed in the previous section. Nevertheless, it is clear that the best performance, robustness, is given by the first approach. However, the best robustness does not imply necessarily the best residuals, in relation with its magnitude.

Table 3.7 – Performance comparison of the observer bank

Observer \ γ	γ_1	γ_2	γ_3
Approach 1	0.0180	0.0152	0.0198
Approach 2	0.6485	0.9656	0.3945
Approach 3	1.76	2.21	1.87

γ_1, γ_2, and γ_3 make reference to the observer $1,2,3$ of the observer bank, respectively.

Three different faults are considered for each sensor, as shown in Fig. 3.9(d). The fault that occurred on the first sensor is a sinusoidal fault, the fault on the second sensor is an abrupt fault, and the fault on the third sensor is an incipient fault. The normalized residual signals are shown in Fig. 3.9(a-c).

For all cases, the fault detection turns out to be successful. The fault isolation is done by comparing the fault signature with the incidence matrix given in Table 3.8. For example, for the fault affecting the sensor 3 all residuals present some changes at $t = 40s$ except the residual r_3 (see colored cells). This particular signature allows to isolate the fault in sensor 3. Similar analysis can be done for the other sensors. In addition, all the observers present good disturbance attenuation, which improves the fault detection. On the other hand, the quality of the residuals is debatable. For example, from Fig. 3.9, it can be seen that the residuals with bigger magnitude are given, by the method presented in the approach 3. Nevertheless, the residuals present also the high level of noise. On the contrary, the smaller residuals are given by the approach 2 but also present the smaller level of noise. From a practical point of view by considering just the magnitude of the residuals, all the methods present useful results. However, by considering also the disturbance rejection, the best method is given by the second approach.

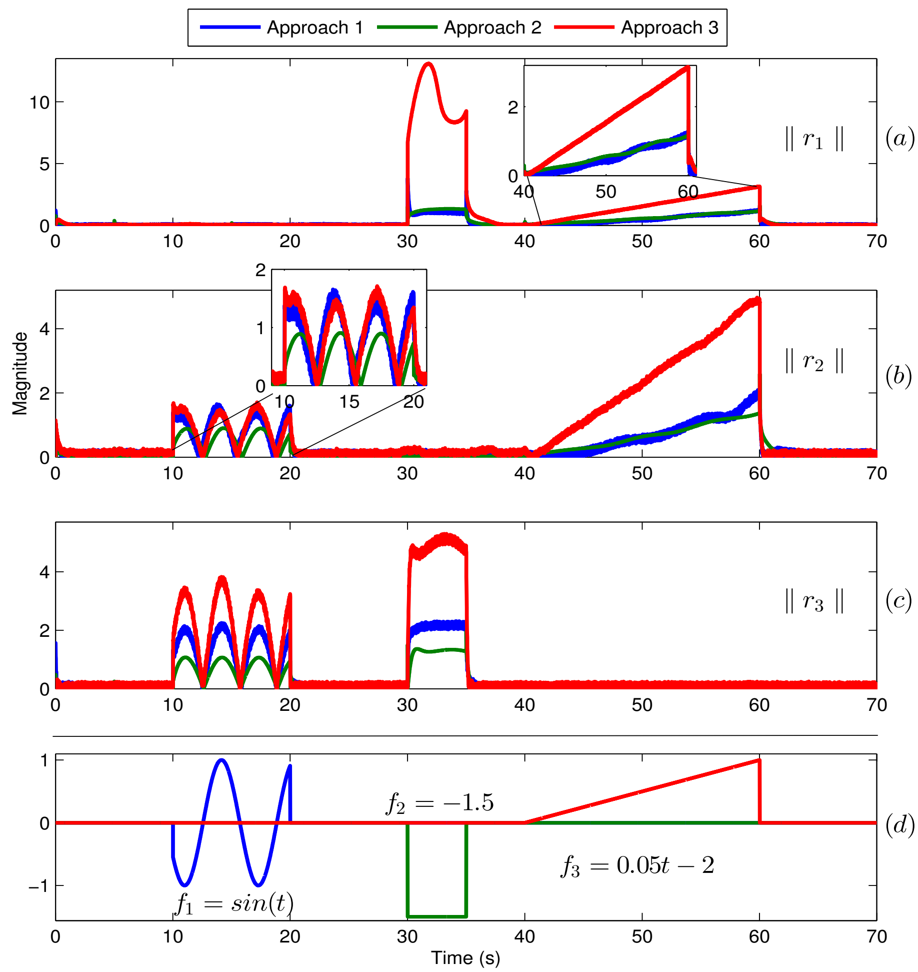


Figure 3.9 – (a)-(c) Residual signals from the observer banks; (d) Sensor faults.

Table 3.8 – Incidence matrix

Residual \ Symptom	S_0	S_1	S_2	S_3
$\| r_1 \|$	0	0	1	1
$\| r_2 \|$	0	1	1	1
$\| r_3 \|$	0	1	1	0

The estimated errors of the gain scheduling functions, between the simulated and the estimated scheduling signals associated with the observer 2, approach 2, are displayed in Fig. 3.10. The remaining es-

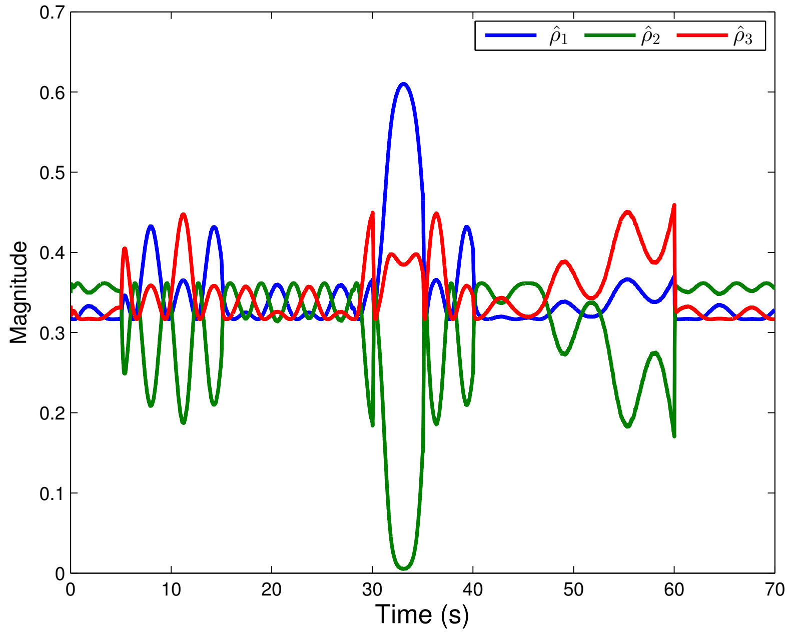


Figure 3.10 – Estimated gain scheduling functions (Approach 2).

timated scheduling functions are not displayed here to avoid redundancy [5]. From Fig. 3.10, it can be deduced that even when the estimated scheduling functions are slightly affected by the faults, the disturbance is well attenuated.

The results presented in this section illustrate the application of the proposed observers to perform fault detection and isolation that are the most important steps in the fault diagnosis process. Nevertheless, the methods do not give any information about the fault magnitude. This problem is addressed in the next section by considering an augmented transformation to estimate states and sensor faults.

3.7 Robust fault estimation based on H_∞ observer

The main contribution of this section is to design a robust state estimation, and fault detection, isolation and estimation based on a standard LPV observer. The results are based on the early work (López-Estrada et al., 2013) with a significant extension by considering additional disturbance rejection to improve the state and fault estimation. To deal with the measurable scheduling problem, this method considers similar transformation as proposed in the Approach 2. Furthermore, in order to perform fault estimation, an

[5] A good estimation of the states for all the approaches implies good estimation of the scheduling functions. Therefore, it is logical that the scheduling functions for the remaining observers have similar shape.

augmented system with faults as auxiliary states on the state vector is considered. Consequently, an observer associated with the uncertain augmented system is synthesized to estimate the original states and fault vectors. The practical contribution of the present section is to apply the proposed method to a realistic model of an anaerobic bioreactor.

Remark 3.7.1. Only Approach 2 was chosen to perform fault diagnosis because it has the best trade-off between disturbance attenuation and optimal solution of the LMIs. Nevertheless, the remaining approaches will be extended in future works.

Consider a descriptor-LPV system under sensor faults and disturbances given by

$$\begin{aligned} \tilde{E}\dot{\tilde{x}}(t) &= \sum_{i=1}^{h} \rho_i(\tilde{x}(t)) \left[\tilde{A}_i\tilde{x}(t) + \tilde{B}_i u(t) + \tilde{B_{di}} d(t)\right] \\ y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}_d d(t) + f(t) \end{aligned} \tag{3.104}$$

where $\tilde{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^l$, $y(t) \in \mathbb{R}^p$, and $f(t) \in \mathbb{R}^p$ are the state vector, the control input, the disturbances vector, the measured vectors and the sensor fault vector, respectively. $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{B}_{di}$, $\tilde{C}$, $\tilde{D}_d$ are constant matrices of appropriate dimensions, $\mathsf{rank}(\tilde{E}) = r \leq n$. h is the number of models and $\rho_i(\tilde{x}(t))$ are the scheduling functions, which are considered to be depending on the unmeasurable state vector $\tilde{x}(t)$. In order to estimate the states and sensor faults, faults are considered as an auxiliary state of the augmented state system $x(t) = \begin{bmatrix}\tilde{x}^T(t) & f^T(t)\end{bmatrix}^T$, such that system (3.104) becomes

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i \left[A_i x(t) + B_i u(t) + B_{di} d(t) + B_f f(t)\right] \\ y(t) &= Cx(t) + D_d d(t) \end{aligned} \tag{3.105}$$

where:

$$\begin{aligned} E &= \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0_p \end{bmatrix}, A_i = \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & -I_p \end{bmatrix}, B_i = \begin{bmatrix} \tilde{B}_i \\ 0_p \end{bmatrix} \\ B_d &= \begin{bmatrix} \tilde{B}_{di} \\ 0 \end{bmatrix}, B_f = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, C = \begin{bmatrix} \tilde{C} & I_p \end{bmatrix}, D_d = \tilde{D}_d. \end{aligned}$$

With the assumptions that the D-LPV system (3.105) is stable and R/I-observable, the following LPV observer is proposed

$$\dot{z}(t) = \sum_{j=1}^{h} \rho_j(\hat{x}(t)) \left[N_j z(t) + G_j u(t) + L_j y(t)\right]$$
$$\hat{x}(t) = z(t) + T_2 y(t) \tag{3.106}$$

where $z(t)$ is an auxiliary state vector, $\hat{x}(t)$ is the state estimation. N_j, G_j, L_j and T_2 are gain matrices of appropriate dimensions. $\rho_j(\hat{x}(t))$ are convex gain scheduling functions which are considered to be depending on the estimated time-varying state $\hat{x}(t)$. Additionally, an auxiliary residual vector is defined as

$$r(t) = y(t) - C\hat{x}(t). \tag{3.107}$$

The problem of fault estimation is reduced to finding the gain matrices of the observer (3.106) which maximize the robustness against the unmeasurable scheduling function $\rho_i(x(t))$ such that $\lim_{t\to\infty} |e(t)| \approx \lim_{t\to\infty} |x(t) - \hat{x}(t)| \approx 0$. In addition, the effects of disturbances must be also attenuated.

Similar to the approach presented in Section 3.3, the system (3.105) is transformed into an uncertain system with estimated gain scheduling function by adding and subtracting the term $\sum_{j=1}^{h} \rho_j(\hat{x}(t)) \left(A_j x(t) + B_j u(t)\right)$ such that the original system (3.105) becomes:

$$E\dot{x}(t) = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left[A_{ij} x(t) + B_{ij} u(t) + B_{di} d(t) + B_f f(t)\right]$$
$$y(t) = Cx(t) + D_d d(t) \tag{3.108}$$

where

$$\sum_{i,j=1}^{h} \rho_i \hat{\rho}_j = \sum_{i=1}^{h} \sum_{j=1}^{h} \rho_i(x(t)) \rho_j(\hat{x}(t)),$$
$$A_{ij} = A_j + \Delta A_{ij}, \ \Delta A_{ij} = A_i - A_j,$$
$$B_{ij} = B_j + \Delta B_{ij}, \ \Delta B_{ij} = B_i - B_j.$$

Note that the previous transformation is possible due to the convex property of the scheduling functions, which considers $\sum_{i=1}^{h} \rho_i(x(t)) = \sum_{i=1}^{h} \rho_j(\hat{x}(t)) = 1$.

The estimation error is given as

$$e(t) = x(t) - \hat{x}(t) \tag{3.109}$$

$$e(t) = (I - T_2C)x(t) - z(t) - T_2D_dd(t), \tag{3.110}$$

assuming that there exists a $T_1 \in \mathbb{R}^{n\times n}$ matrix such that

$$T_1E = I - T_2C. \tag{3.111}$$

A particular solution of matrices T_1 and T_2 is computed as

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger} \tag{3.112}$$

Under this consideration, the error equation is given by

$$e(t) = T_1Ex(t) - z(t) - T_2D_dd(t). \tag{3.113}$$

In order to eliminate the influence of $d(t)$ in (3.113), it is assumed that the unknown inputs have slow variation, i.e $\dot{d}(t) \approx 0$. From a practical point of view, this condition can be relaxed as discussed in (Hamdi et al., 2012b; Chadli et al., 2013b). The dynamics of the error equation is rewritten by considering the slow variation condition as

$$\dot{e}(t) = T_1E\dot{x}(t) - \dot{z}(t) \tag{3.114}$$

and its dynamics is given by

$$\begin{aligned} \dot{e}(t) &= \sum_{i,j=1}^{h} \rho_i\hat{\rho}_j \left[N_je(t) + T_1\Delta A_{ij}x(t) + T_1\Delta B_{ij}u(t) + (T_1B_{di} + K_jD_d)d(t) + T_1B_ff(t)\right] \\ r(t) &= Ce(t) + D_dd(t). \end{aligned} \tag{3.115}$$

Together with the gain synthesis

$$G_j = T_1 B_j \tag{3.116}$$

$$N_j = T_1 A_j + K_j C, \forall j \in [1, 2...h], \tag{3.117}$$

$$K_j = N_j T_2 - L_j, \forall j \in [1, 2...h]. \tag{3.118}$$

Then, a standard representation is obtained by considering the augmented state vector $x_e(t) = \left[e(t)^T \, x(t)^T\right]^T$ such as

$$\begin{aligned} \bar{E}x_e(t) &= \bar{A}x_e(t) + \bar{B}\bar{f}_d(t) \\ r(t) &= \bar{C}x_e(t) + \bar{D}_d \bar{f}_d(t) \end{aligned} \tag{3.119}$$

where

$$\bar{E} = \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \bar{A} = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \begin{bmatrix} N_j & T_1 \Delta A_{ij} \\ 0 & A_i \end{bmatrix}, \bar{B} = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \begin{bmatrix} T_1 \Delta B_{ij} & T_1 B_{di} + K_j D_d & T_1 B_f \\ B_i & B_{di} & B_f \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \bar{D}_d = \begin{bmatrix} 0 & \tilde{D}_d & 0 \end{bmatrix}, \bar{f}_d = [u(t) \, d(t) \, f(t)]^T.$$

Equivalently, the transfer from the input $\bar{f}_d(t)$ to $r(t)$ is written as

$$G_{r\bar{f}_d} = \left\{ \bar{E}, \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D}_d \end{array} \right] \right\}. \tag{3.120}$$

Sufficient conditions to guarantee robustness against unknown inputs and the error provided by the unmeasurable scheduling function such that the norm $\| G_{r\bar{f}_d} \|_\infty \leq \gamma$ are given by the following Theorem:

Theorem 3.7.1. Consider the system (3.104), the augmented system (3.105), and the observer (3.106), and let the attenuation level, $\gamma > 0$. The quadratic stability of the estimation error is guaranteed if $\| G_{r\bar{f}d} \|_\infty < \gamma$ and if there exist matrices $P_1 = P_1^T > 0$, Q_j, $\mathscr{P}_2 = P_2 E + SX$, $X \in \mathbb{R}^{(n-r)\times n}$, $P_2 > 0$, such that the following optimization problem holds $\forall i, j \in [1, 2, ..., h]$:

$$\min_{P_1, P_2, Q_j, X} \gamma$$
$$\text{s.t.}$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & P_1T_1\Delta B_{ij} & \Phi_{14} & P_1T_1B_f & C^T \\ * & \Phi_{22} & \mathscr{P}_2^TB_i & \mathscr{P}_2^TB_{di} & \mathscr{P}_2^TB_f & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & D_d^T \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{3.121}$$

where

$$\begin{aligned} \Phi_{11} &= \mathsf{He}\{(T_1A_j)^TP_1 + Q_jC\}, \\ \Phi_{12} &= P_1(T_1\Delta A_{ij}), \\ \Phi_{14} &= P_1T_1B_d + Q_jD_d, \\ \Phi_{22} &= \mathsf{He}\{A_i^T\mathscr{P}_2\}, \\ \begin{bmatrix} T_1 & T_2 \end{bmatrix} &= \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}. \end{aligned}$$

$S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^TS = 0$.

The observer parameters are computed by $Q_j = P_1^{-1}K_j$ and the conditions defined in (3.116) -(3.118).

Proof. The proof follows the line of the proof of Theorem 3.3.1 by considering the performance criterion

$$J_{r\bar{f}_d} := \int_0^{\infty} r^T(t)r(t)dt - \gamma^2 \int_0^{\infty} \bar{f}_d^T(t)\bar{f}_d(t)dt < 0, \tag{3.122}$$

therefore, it is omitted. □

The following example illustrates the applicability of the proposed method:

Example 3.7.2. Consider the anaerobic bio-reactor from Example 3.3.1 and the state observer

$$\begin{aligned} \dot{z}(t) =& \sum_{j=1}^{16} \rho_i(\hat{\zeta}(t)) \left[N_jz(t) + G_ju(t) + L_jy(t)\right] \\ \hat{x}(t) =& z(t) + T_2y(t). \end{aligned} \tag{3.123}$$

The state observer gain matrices and the attenuation level are computed by solving Corollary 3.7.1 with the YALMIP toolbox (Lofberg, 2004).

Matrices T_1 and T_2 from (3.112)

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Computed matrices from Theorem (3.7.1) (LMI (3.121))

$$P_1 = \begin{bmatrix} 0.3274 & -0.1287 & -0.3218 & 0 & 0.3274 & -0.1287 & -0.3218 \\ -0.1287 & 0.7417 & -0.0919 & 0 & -0.1287 & 0.7417 & -0.0919 \\ -0.3218 & -0.0919 & 0.5486 & 0 & -0.3218 & -0.0919 & 0.5486 \\ 0 & 0 & 0 & 1.5555 & 0 & 0 & 0 \\ 0.3274 & -0.1287 & -0.3218 & 0 & 0.3274 & -0.1287 & -0.3218 \\ -0.1287 & 0.7417 & -0.0919 & 0 & -0.1287 & 0.7417 & -0.0919 \\ -0.3218 & -0.0919 & 0.5486 & 0 & -0.3218 & -0.0919 & 0.5486 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_4 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_6 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$Q_7 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_8 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$Q_9 = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_{10} = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$Q_{11} = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_{12} = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$Q_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.3757 & -3.6067 & -3.6067 & -3.6067 \\ 0 & 0 & 0 & -3.6067 & 2.7577 & 1.2022 & 1.2022 \\ 0 & 0 & 0 & -3.6067 & 1.2022 & 2.7577 & 1.2022 \\ 0 & 0 & 0 & -3.6067 & 1.2022 & 1.2022 & 2.7577 \end{bmatrix}, Q_{14} = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}$$

$$
Q_{15} = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}, Q_{16} = \begin{bmatrix} -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \\ 0 & 0 & 0 & -4.3885 \\ -1.5372 & 0.1267 & 0.7004 & 0 \\ 0.3917 & -2.3716 & 0 & 0 \\ 0.5957 & 0.3702 & -1.9828 & 0 \end{bmatrix}
$$

$$
P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.3757 & -3.6067 & -3.6067 & -3.6067 \\ 0 & 0 & 0 & -3.6067 & 2.7577 & 1.2022 & 1.2022 \\ 0 & 0 & 0 & -3.6067 & 1.2022 & 2.7577 & 1.2022 \\ 0 & 0 & 0 & -3.6067 & 1.2022 & 1.2022 & 2.7577 \end{bmatrix}, S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
X = \begin{bmatrix} 6.000 & -0.0000004 & 1.400 & 9.110 & 9.310 & 9.310\times & 9.310 \end{bmatrix} \times 10^{-04}
$$

The attenuation level is $\gamma = 0.000196$; the small value of γ guarantees good attenuation of uncertainties and disturbances.

For simulation purposes, the state observer initial conditions are $\hat{x}(0) = [0\ 0.5\ 1\ 0.4\ -1\ -.2\ 0.3]^T$. The disturbance signal included in the system is a 0.2 mean random signal bounded by 0.05.

Simulation results are displayed as follows. The interactions between the 16 models defined by the estimated scheduling functions (3.69) are displayed in Fig. 3.11. The quadratic estimation error is displayed in Fig. 3.12. As displayed, the observer converges fast and with a small error. In addition, disturbance attenuation is well performed as initially pointed because the small value of the attenuation level γ. Furthermore, as also displayed, the observer is robust to the unmeasurable gain scheduling functions.

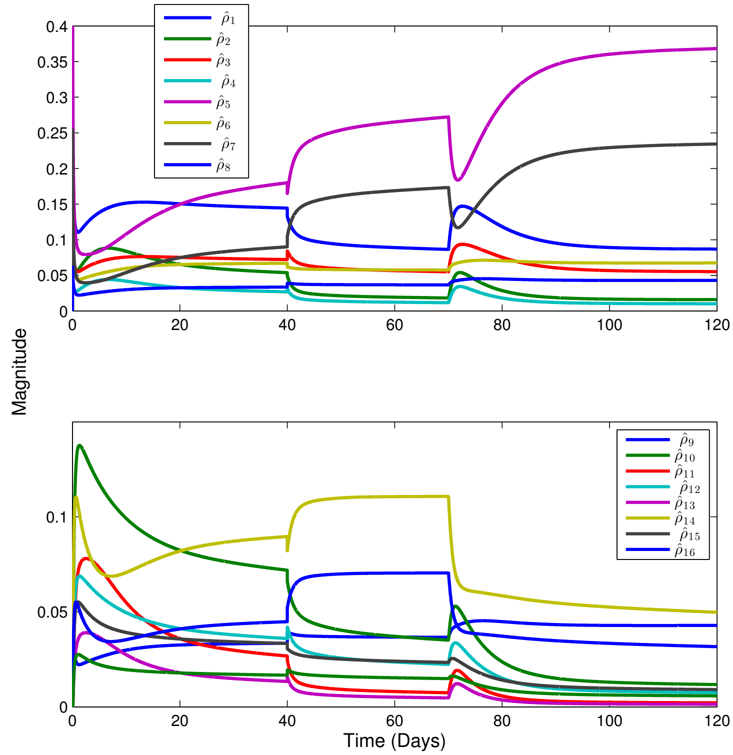


Figure 3.11 – Estimated gain scheduling functions .

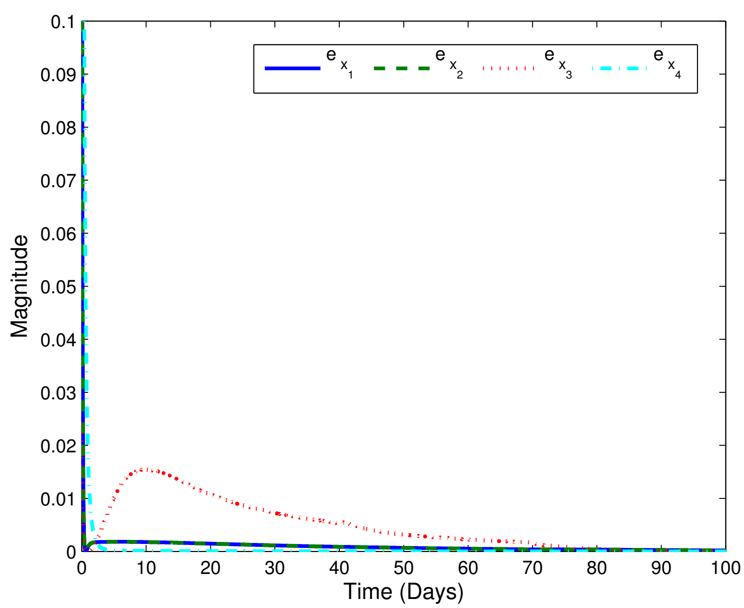


Figure 3.12 – Quadratic estimation errors between the non-linear and the D-LPV model.

The real and the estimated faults are displayed in Fig. 3.13, 3.14, and 3.15 for sensors 1,2 and 3, respectively. The fault on the first sensor is a step-up step-down fault, the fault on the second sensor is an incipient fault and the fault on the third sensor is a step fault. The observer estimates faults with good performance, even when they appear simultaneously. For instance, the RMS errors between f_1 and $\hat{f}_1$, f_2 and $\hat{f}_2$ and, f_3 and $\hat{f}_3$ are 0.0064, 0.0099, and 0.00051, respectively. In addition, it is clear that by considering the H_∞ performance, the error generated by the disturbances and the unmeasurable scheduling functions are well attenuated.

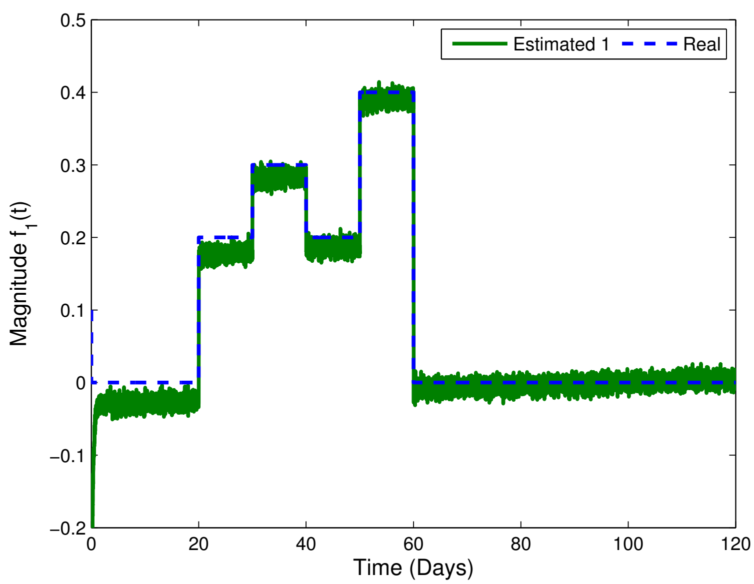


Figure 3.13 – Fault 1 and its estimation.

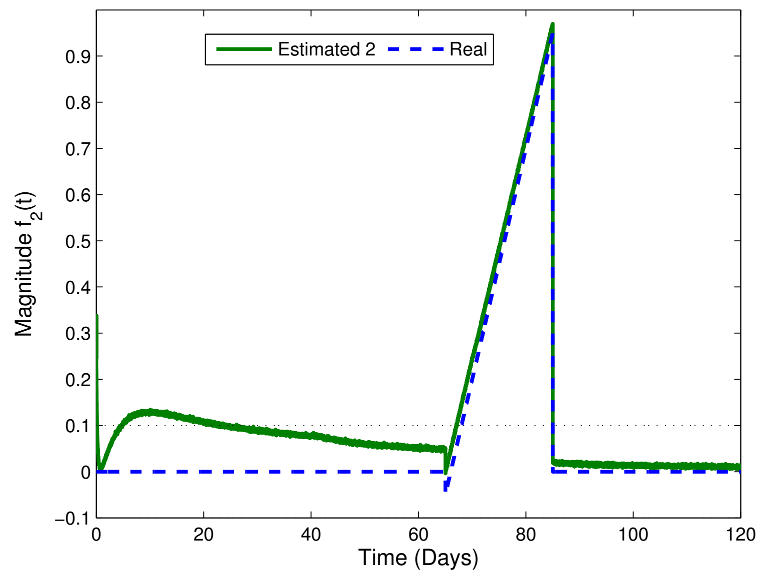


Figure 3.14 – Fault 2 and its estimation.

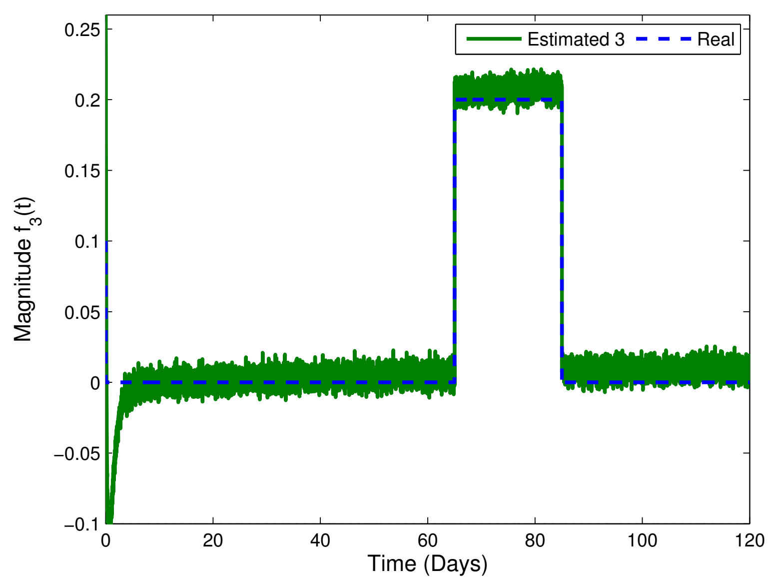


Figure 3.15 – Fault 3 and its estimation.

Moreover, the observer also converges fast (approx 4 days). For example, the fault 3 converges in 3 days. On the other hand, false alarms can be avoided by predefining a small threshold, or different levels of threshold, for each fault or by considering a more elaborated adaptive threshold as in (Montes de Oca et al., 2011). Nevertheless, this problem is beyond the scope of this research.

3.8 Conclusions

As detailed in Chapter 2, prior work has documented the importance of considering the scheduling functions as unmeasurable in order to increase the applicability of the proposed methods. For example, when a D-LPV models is obtained by considering the sector-nonlinearity approach, it has gain scheduling functions that naturally depend on the states. In such case, clearly the typical methodologies are not applicable.

To solve this problem, in this chapter three approaches to design state observers for D-LPV systems with unmeasurable gain scheduling functions have been presented. The first approach has considered a descriptor-LPV observer. The main advantage of this approach is that the solution of the LMIs is "more" optimal than the other two approaches, which is related to its reduced dimension. This can be seen by comparing the magnitude of the attenuation levels, in which case the approach 2 has computed the smallest attenuation level. However, its main drawback is that, as discussed in Chapter 2, a descriptor observer is more sensitive to high frequency noise, which can reduce its applicability, and also disturbances were not considered. These problems could be solved by considering an additional integral gain, which could be addressed in future work. The second approach has studied a standard observer to increase its applicability. As a result, the synthesized solution is less conservative due to the fact that it considers disturbance and noise attenuation and also, as discussed in Chapter 2, a standard observer is less sensitive to high frequency noise. Furthermore, an additional contribution was to illustrate the applicability of the proposed method to a realistic model of an anaerobic bio-reactor. Finally, the third approach has contemplated also a standard observer, but with a different transformation based on the convex property of the scheduling functions. The simulation results show that this approach is "less" optimal than the other two. This could be due to the large size of the LMIs, which could increase the conservatism. Nevertheless, as was illustrated through a common numerical example, that all the proposed methods can perform state-estimation with small error and fast time convergence. Nonetheless, the simulations show that the second approach has the best compromise in terms of disturbance rejection, observer performance, and applicability.

Notably, all the proposed methods can also perform sensor fault detection and isolation, by considering the classical GOS scheme, as was illustrated by a common numerical example. Nevertheless, actuator faults were not considered here and could be addressed in a future work. Moreover, due to the good result given by the second approach, this approach was considered to perform sensor fault diagnosis. Results prove that, by considering the suggested method, it is possible to detect, isolate and estimate small sensor fault magnitudes, despite disturbances and the error generated by the unmeasurable gain scheduling functions, even for simultaneous faults.

Furthermore, our methods were supported by simulations. Future works should address actuator fault detection and isolation and extend the Approaches 1 and 3 to perform fault diagnosis. Moreover, all the methods proposed here were dedicated to continuous-time systems and their extension to discrete-time systems are not straightforward. However, a preliminary solution to the observer design for discrete-time D-LPV systems can be consulted in Appendix C.2.

Finally, the reader must note that all the previous methods can be classified as robust H_∞ fault detection observers, that are robust to disturbances and uncertainties related to the unmeasurable gain scheduling. However, since both disturbances and faults contribute to the residual generated by the fault detection observer, some small faults cannot be detected for a pre-designed threshold. This problem is addressed in the next chapter by considering the H_- index performance on the residual equation. Contrary to the H_∞ approach, H_- methods have gained much attention recently because their main purpose is the study of the worst-case fault sensitivity performance of a fault detection observer. Residuals robust to disturbances, but sensitives to faults can be obtained by combining both theories.

Chapter 4

Fault detection observer design based on the H_-/H_∞ performance

Contents

This chapter is dedicated to the design of a fault detection observer (FDO) for D-LPV systems with unmeasurable gain scheduling functions. The observer design is addressed by considering the H_-/H_∞ approach to maximize the effect of faults on the residuals and minimize the effects of the error caused by the unmeasurable scheduling functions and the effects of the disturbance. Sufficient conditions for the existence of the fault detection observer are established in terms of linear matrix inequalities (LMIs). The chapter is organized as follows: some preliminary and concepts are given in Sec. 4.1. Sec. 4.2 is dedicated to discussing the problem formulation. Sec. 4.3 is dedicated to the fault detection observer design by considering i) H_- fault sensitivity condition, ii) H_∞ robustness against disturbances and noise condition, and iii) the multi-objective problem. A numerical example is presented in Section 4.4.

4.1 Introduction

As discussed in previous chapters, a fault detection system must be able to deal with different types of faults, despite the influence of disturbance. Nevertheless, to guarantee the detection of the worst possible faults, the minimum fault sensitivity must be maximized (Liu and Zhou, 2007). In this context, the H_-/H_∞ index, mainly studied for Linear Time Invariant (LTI) system, is considered to detect faults in the worst-case situation.

H_- index was introduced by (Hou and Patton, 1996) defined as the smallest non-zero singular value of the transfer function matrix from fault to residual at particular frequency ω equal to zero. Sufficient conditions to solve the H_- index given in terms of the LMIs formulation were proposed in (Rambeaux et al., 1999). The H_- index was extended to a more general definition and formulated with Algebraic Riccati Equations (AREs) in (Liu and Wang, 2003; Liu et al., 2005). This extended approach gives a solution to the whole frequency case and does not require that the smallest singular value be non-zero. The inclusion of possible zero singular values in the definition renders the H_- index the worst-case fault sensitivity. Furthermore, H_∞ observers have received much attention in the recent years since they can deal with model uncertainties and disturbances. The H_∞ approach guarantees a solution to disturbance attenuation by considering the worst-case of disturbance. In fact, all the methods proposed in Chapter 3, are dedicated to design state observers with H_∞ disturbance attenuation to guarantee robust state estimation and fault detection by means of residual evaluation. Unlike the previous chapter, this chapter is dedicated to design a fault detection observer by mixing the H_- and the H_∞ approaches.

The main objective of mixed H_-/H_∞ is to find a compromise between fault sensitivity and disturbance attenuation. The idea is to generate robust residuals against system disturbances and sensor noise. Simultaneously, it is desirable to have sensitive residuals when a fault (however minimal) occurs. The problem can be defined as an optimization one. Nevertheless, an optimal sensible optimization criteria could lead to, optimal but, ineffective fault detection filter designs (Liu and Zhou, 2007). Some recent results on this issue for LTI systems are reported in (Henry and Zolghadri, 2005), in which a solution to the multi-objective problem was derived by considering the structured singular value (μ) technique. Later, this method was applied to the electrical flight control system of a civil aircraft in (Henry et al., 2014). Two iterative LMI algorithms, to solve the multi-objective optimization problem for the infinite frequency range case $[0,+\infty)$ and the finite frequency range case $[0, \omega]$ was considered in (Wang et al., 2007). A solution based the GKYP lemma, which provides the possibility to examine the faults and disturbances in the finite-frequency domain, was studied in (Wei and Verhaegen, 2011). Nevertheless, these methods are applicable only to LTI systems and cannot be directly extended to descriptor systems.

To the best of the author's knowledge, there is no work reported for continuous D-LPV systems, which considers the H_-/H_∞ index approach. However, some works on sensor fault detection and diagnosis for continuous standard LPV system, by considering the *descriptor-approach*[1], can be found in (Bouattour et al., 2011; Aouaouda et al., 2013), and for a discrete-time system in Chadli et al. (2013a). Nonetheless, these approaches are not applicable to descriptor systems because, as remarked in Chapter 2, descriptor systems have a more complicate structure and usually have three types of modes (dynamic, impulse, and static) not considered in the descriptor approach. More recently, (Boulkroune et al., 2013) proposed a fault detection observer based on mixed H_-/H_∞ index for a discrete-time descriptor non-linear system, that considers global Lipschitz constraints and Lyapunov stability criterion. Comparing with the previous works, which consider the descriptor, this work considers real descriptor systems which rely on a novel LMI solution. Unlike the work presented in (Boulkroune et al., 2013), the proposed method considers a D-LPV system varying through multiple operating points, while a Lipschitz system remains valid only in a neighborhood of a single nominal operating point. Also, (Boulkroune et al., 2013) considers measurable gain scheduling, while the approach proposed in this thesis considers unmeasurable gain scheduling and continuous D-LPV systems.

In order to illustrate and deduce definitions for observer design with H_-/H_∞, consider the following D-LTI system defined by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + B_d d(t) + B_f f(t) \\ y(t) &= Cx(t) + D_d d(t) + D_f f(t) \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $d \in \mathbb{R}^q$, and $f \in \mathbb{R}^{n_f}$ are the state vector, the control input, the measured output vector, the disturbances, and faults, respectively. E, A, B, B_d, C, D_d, and D_f are constant matrices of appropriate dimensions.

Also consider the the following residual generator

$$\begin{aligned} \dot{z}(t) &= [Nz(t) + Gu(t) + Ly(t)] \\ \hat{x}(t) &= z(t) + T_2 y(t) \\ r(t) &= M\left(y(t) - \hat{y}(t)\right) \end{aligned} \tag{4.2}$$

where $\hat{x} \in \mathbb{R}^n$, $\hat{y}(t) \in \mathbb{R}^p$, and $r(t) \in \mathbb{R}^p$ are the estimated states, the estimated output, and the residuals, respectively. N, G, L, T_2, and M are gain matrices. By connecting the system (4.1) and the observer (4.2), the estimation error is defined as

[1] The descriptor approach consists in transforming a standard system into a descriptor system to estimate sensor faults.

$$
\begin{aligned}
e(t) &= x(t) - \hat{x}(t) \\
e(t) &= (I - T_2C)x(t) - z(t) - T_2D_dd(t) - T_2D_ff(t)
\end{aligned} \tag{4.3}
$$

By assuming $(I - T_2C) = T_1E$, and slow dynamic for $\dot{d}(t) \approx 0$, and $\dot{f}(t) \approx 0$, the dynamic error becomes

$$
\dot{e}(t) = T_1E\dot{x}(t) - \dot{z}(t).
$$

After algebraic manipulations, the following residual error system is obtained

$$
\begin{cases}
\dot{e}(t) &= Ne(t) + (T_1B_d - KD_d)d(t) + (T_1B_f - KD_f)f(t), \\
r(t) &= M(Ce(t) + D_dd(t) + D_df(t))
\end{cases} \tag{4.4}
$$

under the following gain synthesis

$$
\begin{aligned}
G &= T_1B \\
N &= T_1A + KC \\
K &= NT_2 - L \\
\begin{bmatrix} T_1 & T_2 \end{bmatrix} &= \begin{bmatrix} E \\ C \end{bmatrix}.^{\dagger}
\end{aligned}
$$

The residual estimation error system (4.4) is a dynamics system, which states are given by $e(t)$, inputs $d(t)$ and $f(t)$, and output $r(t)$. The Laplace transform of the error can be written as

$$
r(s) = G_{rd}(s) + G_{rf}(s) \tag{4.5}
$$

where

$$
\begin{aligned}
G_{rd}(s) &= MC(sI - N)^{-1}(T_1B_d - KD_d) + MD_d \\
G_{rf}(s) &= MC(sI - N)^{-1}(T_1B_f - KD_f) + MD_f
\end{aligned}
$$

Finally, to guarantee fault sensitivity of the residuals, as possible, and disturbance insensitivity, the fault detection observer (4.2) must be designed such that the residual error system (4.4) converges asymptotically to zero in the fault-free case, and maximize both the robustness against disturbance input $d(t)$ and, in the faulty case, the sensitivity to fault input $f(t)$. To achieve this objective, the residual generator (4.2) can be designed with H_-/H_∞ performance defined as follows:

Definition 4.1.1. Consider the system in (4.1), two scalars β>0 and $\gamma < 0$. The observer (4.2) is called a H_-/H_∞ fault detection observer if

1. The system (4.1) is asymptotically stable
2. $H_- = \| G_{rf}(s) \|_- = \inf_{\omega} \underline{\sigma}(G_{rf}(j\omega)) > \beta$
3. $H_\infty = \| G_{rd}(s) \|_\infty = \sup_{\omega} \bar{\sigma}(G_{rd}(j\omega)) < \gamma$

where $\bar{\sigma}$ and $\underline{\sigma}$ denote the biggest and the smallest singular values of the matrix G_{rf} and G_{rd}, respectively.

Remark 4.1.2. The effect of faults on residual can be measured with H_- index, and the H_∞ norm is used to denote the effect of disturbances.

Remark 4.1.3. The physical meaning of the H_- index is that it measures the smallest amplitude of a transfer matrix from the inputs to the outputs.

By considering the previous definition, the main contribution of this chapter is to consider the design of a robust fault detection observer based on mixed H_-/H_∞ index performance for D-LPV systems. The uncertain system approach, as detailed previously, is considered to minimize the influence of the error provided by the unmeasurable scheduling functions. Based on the uncertain system, the problem is reformulated as an optimization problem to find a trade-off between fault sensitivity and disturbance attenuation. Generalized criterion performances are used to achieve the optimization objectives. Lyapunov equations depending on faults and unknown inputs are considered to guarantee stability. The next sections present in details the main results of the proposed method.

4.2 Problem statement

Consider a continuous descriptor linear parameter varying system affected by unknown inputs and faults as

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{h} \rho_i(x(t)) \left[A_i x(t) + B_i u(t) + B_{di} d(t) + B_{fi} f(t)\right] \\ y(t) &= Cx(t) + D_d d(t) + D_f f(t) \end{aligned} \tag{4.6}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$, $y \in \mathbb{R}^p$, and $f \in \mathbb{R}^{n_f}$ are the state vector, the control input, the disturbances inputs, the measured vector, and the fault vector respectively. A_i, B_i, B_{di}, B_{fi}, C, D_d and D_f are constant matrices of appropriate dimensions.

Remark 4.2.1. Note that, unlike the previous approaches, the matrices B_{fi} and D_f associated with the fault vector $f(t)$ are considered in the D-LPV system (4.6) from the beginning. This consideration is done to obtain an error equation with inputs to $f(t)$ and $d(t)$ and then, maximize the sensitive to $f(t)$ and minimize the sensitive to $d(t)$ on residuals.

Under the assumptions of stability and R- and I-observability, let us consider a fault detection observer for the system (4.6), as

$$\begin{aligned}
\dot{z}(t) =& \sum_{j=1}^{h} \rho_j(\hat{x}(t)) \left[N_j z(t) + G_j u(t) + L_j y(t)\right] \\
\hat{x}(t) =& z(t) + T_2 y(t) \\
r(t) =& M\left(y(t) - \hat{y}(t)\right)
\end{aligned} \tag{4.7}$$

where $z(t) \in \mathbb{R}^n$ represents the state vector of the observer, $\hat{x}(t)$ the estimated state vector. N_j, G_j, L_j, and T_2 are the gain matrices of appropriate dimensions to be synthesized. Note that, unlike the previous approaches, the main problem is to synthesize the observer gains such that the residual effects of disturbances must be minimized while the effects of faults must be maximized, despite the unmeasurable scheduling functions.

A similar procedure as described in Section 3.3 is considered to transform the system with unmeasurable scheduling functions into an uncertain system with estimated gain scheduling functions. Therefore, the system (4.6) is rewritten as an uncertain system by adding and subtracting the term $\sum_{j=1}^{h} \rho_j(\hat{x}(t)) \left(A_j x(t) + B_j u(t)\right)$ to (4.6), such that

$$\begin{aligned}
E\dot{x}(t) &= \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left[\tilde{A}_{ij} x(t) + \tilde{B}_{ij} u(t) + B_{di} d(t) + B_{fi} f(t)\right] \\
y(t) &= Cx(t) + D_d d(t) + D_f f(t)
\end{aligned} \tag{4.8}$$

with

$$\sum_{i,j=1}^{h} \rho_i \hat{\rho}_j = \sum_{i=1}^{h} \sum_{j=1}^{h} \rho_i(x(t)) \rho_j(\hat{x}(t)),\ \forall i,j \in [1,2,...,h]$$
$$\tilde{A}_{ij} = A_j + \Delta A_{ij},\ \Delta A_{ij} = A_i - A_j$$
$$\tilde{B_{ij}} = B_j + \Delta B_{ij},\ \Delta B_{ij} = B_i - B_j$$

From (4.8) with the observer (4.7), the residual estimation error is expressed by

$$\begin{aligned} \dot{e}(t) \quad &= \quad \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left[N_j e(t) + T_1 \Delta A_{ij} x(t) + T_1 \Delta B_{ij} u(t) \right. \\ &\quad \left. + \left(T_1 B_{di} + K_j D_d \right) d(t) + \left(T_1 B_{fi} + K_j D_f \right) f(t) \right] \\ r(t) \quad &= M \quad (y(t) - C\hat{x}(t)) = MCe(t) + MD_d d(t) + MD_f f(t) \end{aligned} \tag{4.9}$$

with

$$G_j = T_1 B_j, \tag{4.10}$$
$$N_j = T_1 A_j + K_j C, \tag{4.11}$$
$$K_j = N_j T_2 - L_j, \tag{4.12}$$
$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}. \tag{4.13}$$

Extending the state vector as $x_e(t) = \begin{bmatrix} e(t)^T & x(t)^T \end{bmatrix}^T$, the augmented residual state-space error system is rewritten as

$$\begin{aligned} \bar{E} x_e(t) &= \bar{A} x_e(t) + \bar{B}_d \bar{d}(t) + \bar{B}_f f(t) \\ r(t) &= \bar{C} x_e(t) + \bar{D}_d \bar{d}(t) + MD_f f(t) \end{aligned} \tag{4.14}$$

with

$$\bar{E} = \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \bar{A} = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \begin{bmatrix} N_j & T_1 \Delta A_{ij} \\ 0 & A_i \end{bmatrix}, \bar{B}_d = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \begin{bmatrix} T_1 \Delta B_{ij} & T_1 B_{di} + K_j D_d \\ B_i & B_{di} \end{bmatrix}$$

$$\bar{B}_f = \begin{bmatrix} T_1 B_{fi} + K_j D_f \\ B_{fi} \end{bmatrix}, \bar{C} = \begin{bmatrix} MC & 0 \end{bmatrix}, \bar{D}_d = \begin{bmatrix} 0 & MD_d \end{bmatrix}, \bar{d}(t) = \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}.$$

The gains of the observer (4.7) are synthesized such that, considering the augmented residual error equation (4.14), the residual will converge asymptotically to zero, in the fault free case, and different from zero in the faulty case. This problem is discussed in detail in the next section.

4.3 Observer gain synthesis

In order to provide a solution to the robust fault detection problem, considering the superposition principle, the residual error equation (4.14) is rewritten in classical formulation as

$$r(t) = G_{rf} f(t) + G_{r\bar{d}} \bar{d}(t) \tag{4.15}$$

where

$$G_{rf} = \left\{ \bar{E}, \left[\begin{array}{c|c} \bar{A} & \bar{B}_f \\ \hline \bar{C} & MD_f \end{array} \right] \right\} \tag{4.16}$$

$$G_{r\bar{d}} = \left\{ \bar{E}, \left[\begin{array}{c|c} \bar{A} & \bar{B}_d \\ \hline \bar{C} & \bar{D}_d \end{array} \right] \right\} \tag{4.17}$$

are the transfer from faults $f(t)$ and inputs $\bar{d}(t)$ to the residuals, respectively. Then, the fault detection observer (4.7) is designed to maximize sensitivity to faults, and also maximize the robustness of the residuals against disturbances. To reach this goal, the multi-objective definition 4.1.1 given for LTI systems is redefined as follows:

Definition 4.3.1. Consider the D-LPV system (4.6), two scalars $\beta > 0$ and $\gamma > 0$. The observer (4.7) is called an H_-/H_∞ fault detection observer if the following conditions are simultaneously fulfilled

$$\| G_{rf} \|_- > \beta \tag{4.18}$$

$$\| G_{r\bar{d}} \|_\infty < \gamma \tag{4.19}$$

Condition (4.18) is used as a measurement of the worst-case fault sensitivity when the system is affected by faults and condition (4.19) denotes the effect of disturbances on residuals. In addition, it is clear that maximizing β and minimizing γ will improve fault detection due to the maximization of effects of faults and minimization of disturbances on residuals. To reach the multi-objective H_-/H_∞ design, in the following sections are considered

- i) the fault sensitivity problem (4.18)
- ii) the robustness condition (4.19)
- iii) the multi-objective problem. (4.18)-(4.19).

4.3.1 Fault sensitivity condition

This section is dedicated to the formulation of conditions to guarantee the maximization of the effects of the faults $f(t)$ on the residual signals $r(t)$. Sufficient conditions to guarantee criteria (4.18), and a constructive method for this design will also be given through the following Theorem:

Theorem 4.3.1. Given system (4.6) and let the attenuation level $\beta > 0$. The robust fault detection observer (4.7) exists if $\| G_{rf} \|_- > \beta$ and if there exist matrices $P_1 = P_1^T > 0$, Q_j, X, and $\mathscr{P}_2 = P_2E + SX$, $P_2 > 0$, such that $\forall i, j \in [1, 2, ..., h]$

$$\begin{bmatrix} C^TM^TMC - \Xi_{11} & -\Xi_{12} & C^TM^TMD_f - \Xi_{13} \\ * & -\Xi_{22} & -\mathscr{P}_2F_i \\ * & * & D_f^TM^TMD_f - \beta^2 I \end{bmatrix} > 0 \tag{4.20}$$

with

$$\begin{aligned} \Xi_{11} &= (T_1A_j)^TP_1 + P_1(T_1A_j)^T + (Q_jC)^T + Q_jC, \\ \Xi_{12} &= P_1T_1\Delta A_{ij}, \\ \Xi_{13} &= P_1T_1B_{f_i} + Q_jD_f, \\ \Xi_{22} &= A_i^T\mathscr{P}_2 + \mathscr{P}_2^TA_i, \\ \begin{bmatrix} T_1 & T_2 \end{bmatrix} &= \begin{bmatrix} E \\ C \end{bmatrix}^\dagger . \end{aligned}$$

$S \in \mathbb{R}^{n\times(n-r)}$ is any full column rank matrix and satisfies $E^T S = 0$.

The observer parameters are computed by $K_j = P_1^{-1} Q_j$ and the equations defined in (4.10)-(4.12).

Proof. Condition (4.18) of G_{rf} considers the energy transfer from faults $f(t)$ to the residual output $r(t)$, which is greater than β under zero initial conditions if

$$J_{rf} := \int_0^\infty r^T(t) r(t) dt - \beta^2 \int_0^\infty f^T(t) f(t) dt > 0 \tag{4.21}$$

To guarantee asymptotic stability, a candidate Lyapunov function is considered as $V(x_e(t)) = x_e^T(t)\bar{E}^T P x_e(t)$, such that

$$\begin{aligned}
\int_0^\infty \left(r^T(t) r(t) d\tau - \beta^2 f^T(t) f(t) - \dot{V}(x_e(t))\right) d\tau + V(x_e(t)) > 0 \\
\int_0^\infty \left((MCe(t) + MD_f f(t))^T (MCe(t) + MD_f f(t)) - \beta^2 f^T(t) f(t) - \dot{V}(x_e(t))\right) d\tau + V(x_e(t)) > 0 \\
\int_0^\infty \left(e^T(t)\left(C^T M^T MC\right) e(t) + e^T(t)\left(C^T M^T MD_f\right) f(t) + f^T(t)\left(D_f^T M^T MC\right) e(t)\right. \\
\left. + f^T(t)\left(D_f^T M^T MD_f\right) f(t) - \beta^2 f^T(t) f(t) - \dot{V}(x_e(t))\right) d\tau + V(x_e(t)) > 0 \\
\int_0^\infty \left\{ \begin{bmatrix} e(t) \\ x(t) \\ f(t) \end{bmatrix}^T \begin{bmatrix} C^T M^T MC & 0 & C^T M^T MD_f \\ * & 0 & 0 \\ * & * & D_f^T M^T MD_f - \beta^2 I \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \\ f(t) \end{bmatrix} - \dot{V}(x_e(t)) \right\} d\tau + V(x_e(t)) > 0
\end{aligned}$$

Considering that $\bar{E}^T P = P^T \bar{E}$, the Lyapunov function is computed as

$$\begin{aligned}
\dot{V}(x_e(t)) &= \dot{x}_e^T(t)\bar{E}^T P x_e(t) + x_e^T \bar{E}^T P \dot{x}_e(t) \\
\dot{V}(x_e(t)) &= \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left[x_e^T(t)\bar{A} P x_e(t) + f^T(t)\bar{B_f}^T P x_e(t) + x_e^T(t) P^T \bar{A} x_e(t) + x_e^T(t) P^T \bar{B}_f f(t) \right] \\
\dot{V}(x_e(t)) &= \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left\{ \begin{bmatrix} x_e \\ f(t) \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P + P^T \bar{A} & P^T \bar{B}_f \\ * & 0 \end{bmatrix} \begin{bmatrix} x_e \\ f(t) \end{bmatrix} \right\}
\end{aligned}$$

Substituting $\bar{A}$, $\bar{B}_f$ from (4.14), $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, with $P_1 = P_1^T > 0$ and $E^T P_2 = P_2^T E$, it results

$$\dot{V}(x_e(t)) = \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left\{ \begin{bmatrix} e(t) \\ x(t) \\ f(t) \end{bmatrix}^T \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & P_2 B_{fi} \\ * & * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \\ f(t) \end{bmatrix} \right\}$$

The performance criterion is rewritten as

$$J_{rf} := \int_0^\infty \sum_{i,j=1}^{h} \rho_i \hat{\rho}_j \left\{ \begin{bmatrix} e(t) \\ x(t) \\ f(t) \end{bmatrix}^T \Upsilon \begin{bmatrix} e(t) \\ x(t) \\ f(t) \end{bmatrix} \right\} dt + V(x_e(t)) > 0 \tag{4.22}$$

with

$$\Upsilon := \begin{bmatrix} C^T M^T M C - \Xi_{11} & -\Xi_{12} & C^T M^T M D_f - \Xi_{13} \\ * & -\Xi_{22} & -P_2 B_{fi} \\ * & * & D_f^T M^T M D_f - \beta^2 I \end{bmatrix} > 0.$$

Hence, if $\bar{E}^T P = P^T \bar{E}$ and $\Upsilon > 0$ implies $J_{rf} > 0$. The equality constraint can be eliminated by considering Lemma 3.2.2. As a result, the LMI (4.20) is obtained.This ends the proof. □

4.3.2 Robustness condition

This section is dedicated to the formulation of conditions to ensure the maximum attenuation of effects of inputs $\bar{d}(t)$ on the residual signals $r(t)$. The following Theorem provides sufficient conditions to guarantee the criteria (4.19).

Theorem 4.3.2. Given system (4.6) and let the attenuation level $\gamma > 0$. Then the robust fault detection observer (4.7) exists if $\| G_{r\bar{d}} \|_\infty < \gamma$ and if there exist matrices $P_1 = P_1^T > 0$, Q_j, X, and $\mathscr{P}_2 = P_2 E + SX$, $P_2 > 0$ $\forall i, j \in [1, 2, ..., h]$, such that

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & P_1 T_1 \Delta B_{ij} & \Xi_{14} & (MC)^T \\ * & \Xi_{22} & \mathscr{P}_2 B_i & \mathscr{P}_2 B_{di} & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\gamma^2 I & (MD_d)^T \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{4.23}$$

with $\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}$, Ξ_{11}, Ξ_{12}, Ξ_{22} as defined as in (4.20), and $\Xi_{14} = P_1 T_1 B_d + Q_j D_d$. $S \in \mathbb{R}^{n\times(n-r)}$ is any full column rank matrix and satisfies $E^T S = 0$.

The observer parameters are computed as $K_j = P_1^{-1} Q_j$ and the equations defined in (4.10)-(4.12).

Proof. The Proof follows the line of Theorem 3.3.1 that considers the performance criterion

$$J_{r\bar{d}} := \| r(t) \|_2^2 \leq \gamma^2 \| \bar{d}(t) \|_2^2 . \tag{4.24}$$

as detailed in Chapter 3, therefore is omitted. □

4.3.3 Mixed H_-/H_∞ observer design

Theorems 4.3.1 and 4.3.2 are combined to guarantee the maximum sensitivity of faults $f(t)$ and the maximum attenuation effects of inputs $\bar{d}(t)$ on the residual $r(t)$. This combination provides sufficient conditions to satisfy criteria (4.18) and (4.19) .

Theorem 4.3.3. Given the system (4.6) and let the attenuation levels $\gamma > 0$ and $\beta > 0$. The robust fault detection observer (4.7) exists and satisfies conditions (4.18) and (4.19) if there exist matrices $P_1 = P_1^T > 0$, Q_j, X, and $\mathscr{P}_2 = P_2 E + SX$, $P_2 > 0$, $\forall i, j \in [1, 2, ..., h]$, such that

$$\begin{bmatrix} C^T M^T MC - \Xi_{11} & -\Xi_{12} & C^T M^T MD_f - \Xi_{13} \\ * & -\Xi_{22} & -\mathscr{P}_2 F_i \\ * & * & D_f^T M^T MD_f - \beta^2 I \end{bmatrix} > 0 \tag{4.25}$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & P_1 T_1 \Delta B_{ij} & \Xi_{14} & (MC)^T \\ * & \Xi_{22} & \mathscr{P}_2 B_i & \mathscr{P}_2 B_{di} & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\gamma^2 I & (MD_d)^T \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{4.26}$$

where $\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger}$. $S \in \mathbb{R}^{n\times(n-r)}$ is any full column rank matrix and satisfies $E^T S = 0$.

The observer parameters are computed as $K_j = P_1^{-1} Q_j$ and the equations defined in (4.10)-(4.12).

Theorem 4.3.3 combines the maximum sensitivity of faults $f(t)$ and the maximum attenuation of disturbances on the residual $r(t)$, such that robustness against disturbance is guaranteed by solving the LMI (4.26), while the LMI (4.25) guarantee the maximum sensitivity to faults. Nevertheless, it must be noted that to guarantee a solution to the H_{-} index problem, the term $D_f^T D_f - \beta^2 I > 0$ must be bigger than zero, otherwise the LMI system becomes infeasible.

4.4 Illustrative Example

This section is dedicated to analyze a numerical example in order to illustrate the effectiveness of the proposed method. Three simulations are proposed to analyze the observer performance in fault-free operation, with a constant fault, and frequency faults. For such purpose, let us consider the descriptor system (4.6), whose state matrices are given by

$$E = \mathrm{diag}(1,1,0), A_1 = \begin{bmatrix} -10 & 5 & 6.5 \\ 2 & -5.5 & -1.25 \\ -9 & 4 & 8.5 \end{bmatrix}, A_2 = \begin{bmatrix} -10 & 5 & 6.5 \\ 5 & -4 & -1.25 \\ -2 & 4 & 7 \end{bmatrix}, A_3 = \begin{bmatrix} -8 & 5 & 6.5 \\ 5 & -4 & -1.25 \\ -5 & 4 & 6 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.7 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 0.5 \\ 0.6 \end{bmatrix}, B_{d1} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}, B_{d2} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}, B_{d3} = \begin{bmatrix} 0.2 \\ 0 \\ 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, D_d = \begin{bmatrix} 0.5 \\ 0.3 \\ 0 \end{bmatrix}, D_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The scheduling functions are the same than the Example 3.2.2, rewritten here for convenience

$$\begin{aligned} \rho_i(x(t)) = & \frac{\mu_i(x(t))}{\sum_{i=1}^{3} \mu_i(x(t))}, \forall i \in [1,2,3], \\ \mu_1(x(t)) = & \quad exp\left[\frac{1}{2}\left(\frac{x_1(t)+0.4}{0.5}\right)^2\right] \\ \mu_2(x(t)) = & \quad exp\left[\frac{1}{2}\left(\frac{x_1(t)-0.4}{0.1}\right)^2\right] \\ \mu_3(x(t)) = & \quad exp\left[\frac{1}{2}\left(\frac{x_1(t)-1}{0.5}\right)^2\right]. \end{aligned} \tag{4.27}$$

In order to solve the LMI set (4.25)-(4.26), the following iterative algorithm is proposed

Algorithme 4.1 Algorithm to solve the multi-objective H_-/H_∞ problem.

- Step 1: Choose a sufficient high value $\beta(0) > 0$ such that the LMIs system is infeasible.
- Step 2: Solve the LMIs system by considering the following optimization problem: $min(\gamma)$ subject to (4.25) and (4.26).
- Step 3: If the system is infeasible, then decrease β step by step.
- Step 4: Repeat step 2 until a feasible solution is found.
- Step 5: Compute the gain matrices $K_j = P_1^{-1}Q_j$, N_j, L_j, and G_j.

Note that, despite the decreasing value of β, Algorithm 4.1 computes the minimum γ for a given value β such that the LMI set has a solution. In other words, Algorithm 4.1 concerns with maximizing β and minimizing γ such the LMIs of Theorem 4.3.3 are feasible. It is equivalent on maximize fault sensitivity and also maximize robustness to disturbances. The computed solution is suboptimal due to the constraints imposed to some variables and matrices, and also due to the new matrices used to avoid the equality constrains $\bar{E}^T P = P^T \bar{E}$. Nevertheless, the iterative algorithm is optimized to compute the gains of the observer which can generate useful residual signals. Investigation of the optimal solution is beyond the scope of this Thesis and it remains an open problem for future research.

Algorithm 4.1 was implemented with YALMIP toolbox (Lofberg, 2004) and SeDuMi solver (Sturm, 1998). The weighting matrix was chosen as $M = I$, in order to get LMI conditions. Matrices T_1 and T_2 computed from (4.13) are

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6667 & 0 \\ -1 & -0.333 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} -0.00001 & -0.00001 \\ 0.3333 & -0.3333 \\ 0.3333 & 0.6667 \end{bmatrix}$$

The proposed Algorithm 4.1 starts[2] with $\beta(0) = 4$, and decreasing β by 0.01, the solution is found with $\beta = 0.9950$ and $\gamma = 1.160$. As remarked before, the solution is sub-optimal, with the understanding that an optimal solution implies $0 < \gamma < 1$ and $\beta > 1$. Nevertheless, it will be demonstrated that with the computed solution, it is possible to perform fault detection. The observer parameters are:

[2]The reader must note that there is not a general rule to select the starting value of β. However, if the starting value of β is too high, then the computation can increase exponentially. The best is to select the smaller value of β for in which case the LMI set is infeasible.

$$P_1 = \begin{bmatrix} 0.7319 & 0.1713 & 0.6027 \\ * & 0.4969 & 0.0248 \\ * & * & 0.6247 \end{bmatrix}, Q_1 = \begin{bmatrix} -1.5306 & 0.1067 \\ -0.6093 & -0.4492 \\ -1.5424 & 0.1343 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -1.1342 & 0.4272 \\ -0.7145 & -0.3467 \\ -1.5371 & 0.2009 \end{bmatrix}, Q_3 = \begin{bmatrix} -1.2201 & 0.4143 \\ -0.7364 & -0.3354 \\ -1.5300 & 0.2174 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.8747 & -0.0016 & -0.0007 \\ * & 1.0000 & 0 \\ * & * & 2.2095 \end{bmatrix}, X = \begin{bmatrix} -1.4878 & 0.1445 & -0.4086 \end{bmatrix}.$$

The gain matrices are:

$$N_1 = \begin{bmatrix} -7.8169 & 6.1302 & 8.6831 \\ -1.3374 & -5.1079 & -3.5041 \\ 5.0790 & -6.6689 & -10.3377 \end{bmatrix}, N_2 = \begin{bmatrix} -1.7237 & 10.0974 & 14.7763 \\ -1.1600 & -5.4998 & -5.3266 \\ -1.6127 & -10.9330 & -16.0293 \end{bmatrix},$$

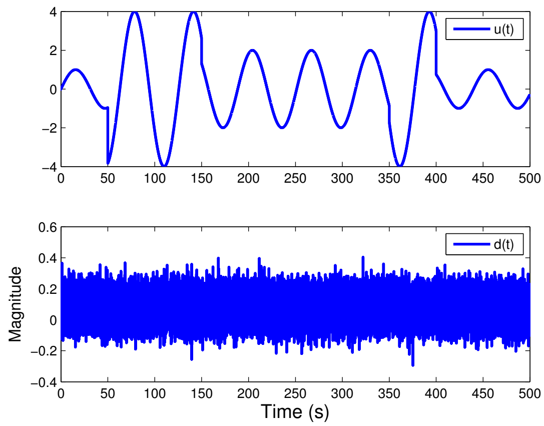
$$N_3 = \begin{bmatrix} -0.8326 & 9.2886 & 13.6674 \\ -0.8535 & -5.3041 & -5.0202 \\ -2.5172 & -10.1491 & -14.9338 \end{bmatrix}, L_1 = \begin{bmatrix} 3.8076 & 2.6924 \\ -1.4295 & 0.5961 \\ -2.1666 & -3.9167 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 3.1938 & 3.3062 \\ -0.7757 & -0.0577 \\ -1.7211 & -4.3622 \end{bmatrix}, L_3 = \begin{bmatrix} 3.3634 & 3.1366 \\ -0.8040 & -0.0294 \\ -1.8786 & -4.2048 \end{bmatrix}.$$

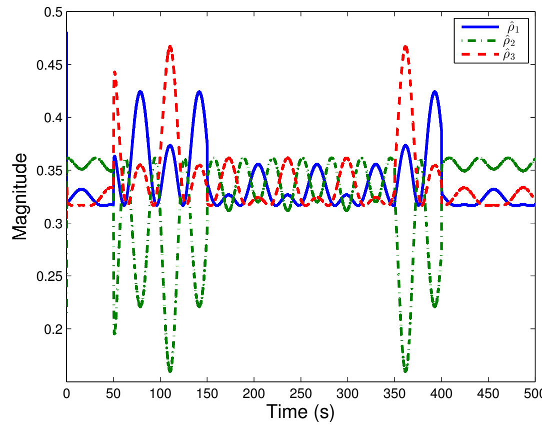
Then, by considering the computed gain matrices, three simulation scenarios are analyzed below.

Simulation under fault-free operation

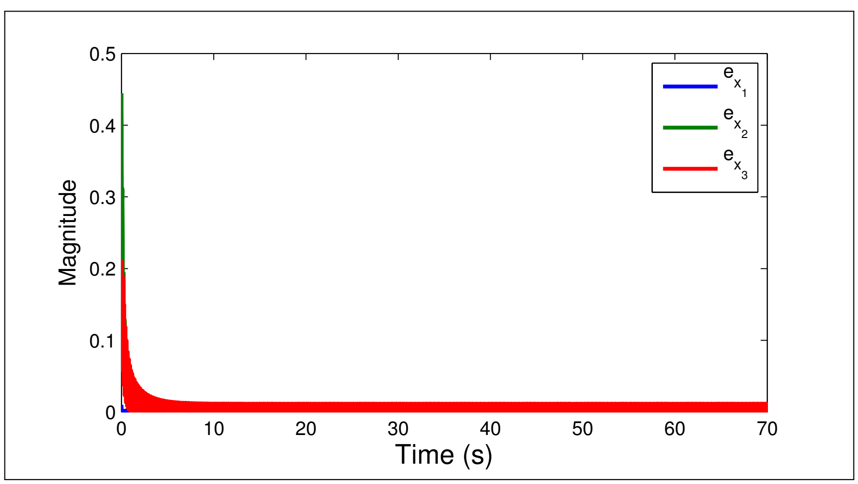
The objective of the first simulation is to show the observer performance under fault-free operation and test the robustness against disturbance/noise. For such purpose, simulation conditions are considered as follows: $x(0) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ and $\hat{x}(0) = \begin{bmatrix} 0.1 & -2 & -3 \end{bmatrix}^T$. A sinusoidal input $u(t)$ is applied, as displayed in Fig. 4.1a. The included disturbance $d(t)$ is a random Gaussian noise with zero mean and variance 0.2, as shown in Fig. 4.1a.



(a) Applied input and noise



(b) Estimated gain scheduling functions



(c) Estimation errors e_{x_i}

Figure 4.1 – Observer performance in fault-free operation.

The results of the first simulation are analyzed as follows: as displayed in Fig. 4.1c, the robust observer converges fast, asymptotically, and with a small error. In addition, by comparing the disturbance magnitude and the small estimation error, it is clear that the disturbance is well attenuated. The estimated gain scheduling functions are displayed in Fig. 4.1b. Both results illustrate good estimation of states and gain scheduling functions. In other words, the results validate the effectiveness of the proposed method in fault-free operation. The next simulation tests the observer performance under a sensor fault.

Simulation under a step fault

The objective of the second simulation is to illustrate the observer performance under a constant fault. In this case, same simulation conditions, as given in the first simulation, are considered. A fault in sensor 2, from $50 \leq t \leq 450$ with magnitude 0.2, is introduced. Note that the magnitude of the fault is less than the magnitude of the disturbance in order to test the observer performance under fault/disturbance with similar magnitudes.

The evolution of the normalized-residual evaluation and faults are displayed in Fig. 4.2, in which the solid lines represent the residuals associated with the outputs and the dashed line stands for the fault. The results show that $\parallel r(t)_{1,2} \parallel \approx 0.11$ in fault free operation, for $t < 50$ and $t > 450$. Nevertheless, when the fault occurs from $50\text{s} \leq t \leq 450\text{s}$, the residual $\parallel r(t)_2 \parallel \approx 0.34$ while the residual $\parallel r(t)_1 \parallel$ remains without any change. This means that the fault in sensor 2 can be detected and isolated after its occurrence. Additionally, false alarms can be easily avoided by selecting a predefined threshold. The next simulation shows the observer performance under frequency faults.

The best and the worst-case fault sensitivity

The third simulation has the objective of illustrating the observer performance under a frequency varying fault to identify the worst-case fault sensitive. For linear systems, the worst and the best fault sensitivity points can be identified with Bode or singular value plots. Nevertheless, D-LPV systems are time-varying, and hence frequency responses plots cannot be used. For such purpose, a frequency-time varying fault, in sensor 2, is considered. The magnitude of the fault is $0.2\sin(\omega t)$ inserted from $50s \leq t \leq 450s$. $500s$ of simulation time was selected in order to see at least one period of the sinusoidal fault at small frequencies. To identify the worst-fault sensitivity, 50 simulations were done by varying logarithmically the frequency of the fault from $\omega = 0.01$ to $\omega = 1 \times 10^5$ rad/s. The simulation conditions, input and noise, are the same that in the previous example.

With the objective to link the computed data to a typical frequency response plot, the computed points were fitted with the Curve Fitting Toolbox. The results are displayed in Fig. 4.3a, in which the black

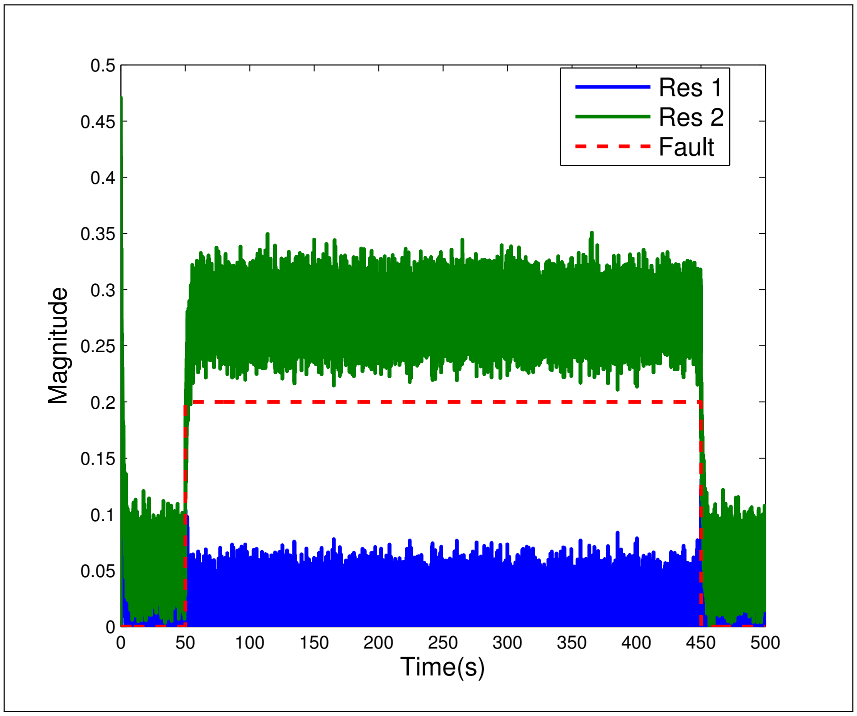
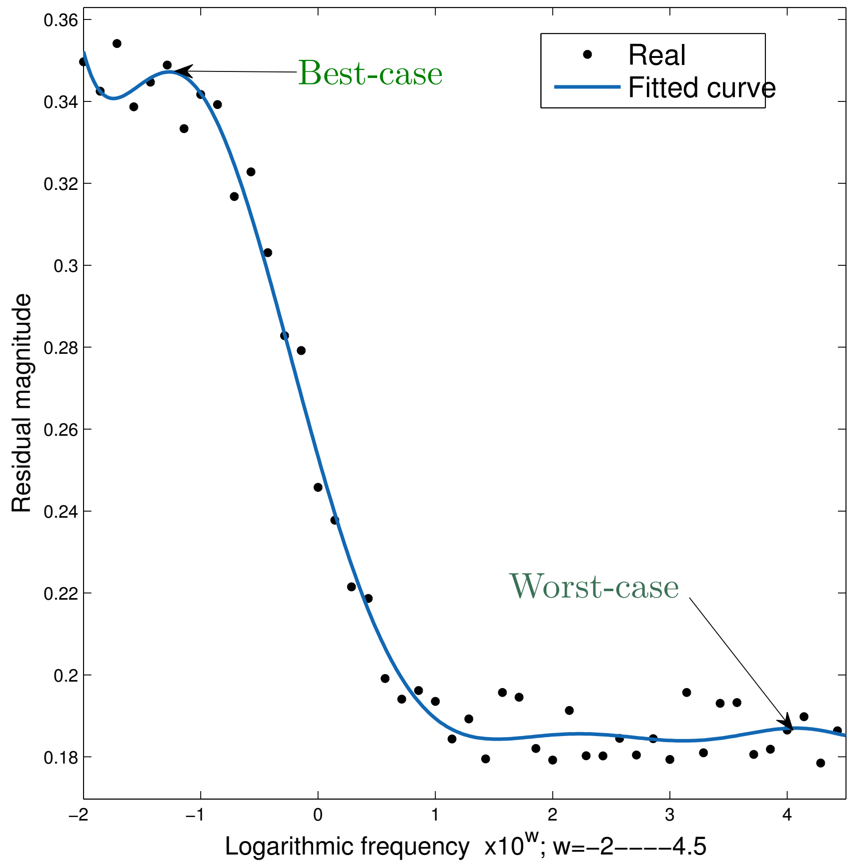


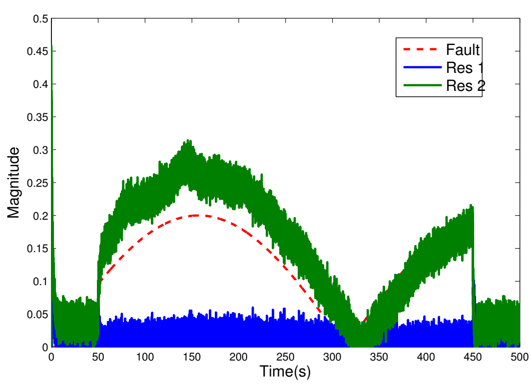
Figure 4.2 – Residuals signals and fault

points represent the maximum residual magnitude at each ω. The curve fitting is represented by the solid line. For both the real and the fitted data, the abscissa coordinate represents the frequency variation in logarithmic scale and the ordinate axis represents the gain magnitude of the residual.

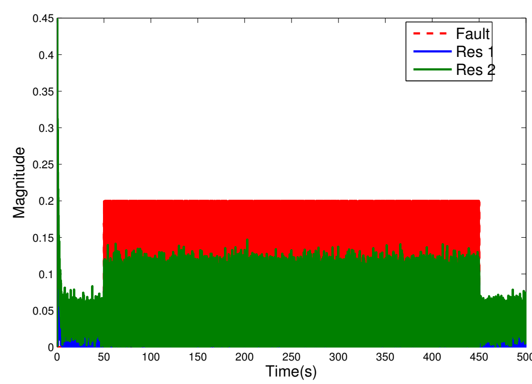
From Fig. 4.3a, it is easy to identify that the best-case fault sensitivity is obtained at small frequencies, e.g. from $\omega = 0.01$ rad/s to $\omega = 1$ rad/s with $\| r(t)_2 \| \approx 0.35$. The worst-fault sensitivity case is presented at high frequencies $\omega > 100$ rad/s with $\| r(t)_2 \| \approx 0.15$. The simulations corresponding to the best and the worst-case fault sensitivity are shown in Fig. 4.3b-4.3c, with the solid line representing the residuals, and the dashed line the fault. As displayed in 4.3b, in the best-case the effect of the fault on $\| r(t)_2 \|$ is well amplified, which facilitates the detection. As displayed in Fig. 4.3c, in the worst-case, the effect of the fault on $\| r(t)_2 \|$ is less amplified. Nevertheless, in both cases, the FDO is always sensitive, despite the faults being small, and it is always possible to detect the fault by considering a small predefined threshold, regardless of disturbances, $d(t) > f(t)$, and the error generated by the unmeasurable scheduling functions. The results validate the effectiveness of the proposed method.



(a) Gain evolution over frequency



(b) The best-case fault sensitivity



(c) The worst-case fault sensitivity

Figure 4.3 – Identification of the best and the worst case fault sensitivity.

4.5 Conclusions

This chapter was dedicated to obtaining sufficient conditions to design a fault detection observer with H_{-}/H_{∞} performance. The objective was to generate residual signals sensitive as possible to faults, but insensitive to disturbances. To achieve this objective, sufficient conditions to synthesize the observer gains were proposed, as a set of LMIs. To solve the LMI set, an algorithm was considered to compute the observer gains subject to find a compromise between fault sensitivity and disturbance attenuation on the residuals.

The results outlined in this Chapter demonstrate the effectiveness of the proposed method to estimate states in fault-free operation and to detect and isolate faults. The reader must consider that for all simulations, the magnitude of the fault was less than the magnitude of the disturbance, to illustrate the effectiveness of both fault amplification and disturbance attenuation. The best and the worst case fault sensitivity were identified by considering a sinusoidal frequency varying fault with constant amplitude. As a result for the proposed example, the best-case fault sensitivity was found at small frequencies and the worst at high frequencies. The simulations show that even in the worst case, the proposed method can detect and isolate faults by considering a small predefined threshold. Nevertheless, this approach is inherently limited in its precision to find an optimal solution to the LMI set. This problem could be solved by considering a limited frequency response, but this problem is beyond the scope of this research and is a topic for future work. Nonetheless, it is well known that in practice, the most common sensor fault is offset bias that, alters the sensor readings uniformly by a certain value. In that case, if the results of the numerical example are extrapolated to sensor bias fault, then it is clear that the proposed approach can detect and isolate this kind of fault regardless of its magnitude.

Nevertheless, it is encouraging that, despite the limitations, this approach can produce residuals with an acceptable degree of accuracy, regardless of the disturbance and the unmeasurable scheduling functions, that validates the initial objective of the approach.

Chapter 5

Conclusions and perspectives

This thesis considered different methods for fault detection and isolation based on an observer design for descriptor-linear parameter varying (D-LPV) systems. This chapter summarizes the work presented in this thesis to review the main conclusions.

D-LPV systems can incorporate and describe nonlinear behavior and time varying aspects. Moreover, the descriptor part can include slow dynamics expressed through ordinary differential equations and static dynamics expressed through algebraic equations. All these properties made D-LPV systems mathematical models that can represent, or approximate to an arbitrary degree of accuracy, a large class of nonlinear systems with a compact set of descriptor-linear time invariant (D-LTI) models. However, compared with the traditional approach, which considers the scheduling functions as depending on a measurable scheduling parameter, the methods proposed in this thesis consider the scheduling function as depending on the unmeasurable state vector.

General conclusions can be enumerated based on the accomplishments of this research:

- The basic concepts of model-based fault detections were explained briefly in **Chapter 1**. This brief explanation considers the three steps to perform fault detection and isolation based on residual generation. This chapter also presents the main objectives of the research and a summary of the main contributions.

- **Chapter 2** was divided in five parts. The first part presented a background of D-LPV system modeling. In the second part, it was given a detailed review of standard and D-LPV systems. The review was organized in two sections in order to illustrate the similarities and common ideas of systems with measurable and unmeasurable gain scheduling functions. The third part was dedicated to D-LPV modeling methods. Detailed calculation of D-LPV models based on descriptor nonlinear equations were studied by considering the linearization and the sector nonlinearity methods. Three different models were proposed to demonstrate the modelization methods. The fourth part was dedicated to the study of the basic concepts of D-LPV systems. First, stability property is analyzed by means of standard pole allocation and equivalent transformations, and then by LMI descriptions obtained with quadratic Lyapunov equations. In addition, observability assumptions were given by considering the direct method related to triple (E, A_i, C) matrices. The fifth part of Chapter 2 was committed to the observer design of D-LPV systems with measurable and unmeasurable scheduling functions. The goal of this part was to exemplify the methodological differences related to the observer design. For example, the observer synthesis of the D-LPV systems with measurable scheduling functions can be addressed by considering standard Lyapunov equations, while the observer synthesis of systems with unmeasurable scheduling functions need stability conditions and also robustness conditions to minimize the error provided by the unmeasurable scheduling vector. The literature shows that the best method to address this problem is considering the H_∞ formulation. Finally, the last part of the chapter was dedicated to presenting some numerical algorithms that compose a toolbox that can be auxiliary for observer design and fault diagnosis of descriptor systems.

- **Chapter 3** gather four of the main results of this thesis related to the robust state observer design and its application to fault detection and isolation. The H_∞ approach was considered in the observer design to guarantee stability on of the estimation error, disturbance attenuation, and also to minimize the error given by the unmeasurable scheduling functions. As a result, the first part of the chapter presented three different approaches to design state observers for D-LPV systems. These approaches considered different transformations in order to represent the error between the unmeasurable and the estimated scheduling functions as model uncertainties. Furthermore, the first approach was dedicated to the design of a descriptor-LPV state observer, while the second and third approaches were dedicated to the design of a standard-LPV observer. For all approaches, sufficient conditions to guarantee the asymptotic convergence of the estimated states with respect to the real states and H_∞ performance were obtained in terms of Linear Matrix Inequalities (LMIs). Simulation examples were proposed to illustrate the effectiveness of the methods. The simulation results show that the third method guaranteed better robustness and stability conditions when noise was not considered. However, when noise was considered, the second approach gave better results. This change in the observer performance was related to the dimension of the LMIs, and the higher number of unknown matrices, which are greater when a noise matrix is considered, and increased the conservatism of the numerical solution of the third approach. Moreover, the simulations showed that the three methods provided useful residuals that were necessary to perform successfully fault detection and isolation, as illustrated in the second part of the Chapter. In addition, the practical contribution of this part was to apply the second approach to a realistic model of an anaerobic bioreactor.

The third part of the chapter used the results of the second approach to design a robust jointly sensor and state estimation observer with H_∞ performance. In order to estimate sensor faults, the states of the D-LPV system were augmented by considering the fault vector as an auxiliary state variables. Sufficient conditions for the existence of the robust observer were given by a set of linear matrix inequalities. The method was tested by considering the simulation of the anaerobic bioreactor. The results showed that the observer could estimate the states and the faults with a small error and fast convergence; even in the case of simultaneous faults.

- **Chapter 4** addressed the design of a standard observer with H_-/H_∞ performance. Different from the results obtained by the previous approaches, which can be seen as a generalization of the worst disturbance case attenuations, the inclusion of H_- performance guaranteed the design of an observer used to generate residuals sensitive as possible to faults and insensitive to disturbances. This was equivalent to maximize fault sensitivity, and also maximize robustness to disturbances. However, the computed solution was suboptimal due to the constraints imposed to some variables and matrices. Research of an optimal solution is beyond the scope of this thesis and it remains an open problem for future research. Simulation results showed that the proposed fault detection observer satisfied the multi-objective performance, despite the unmeasurable scheduling-parameter and disturbances.

Based on the previous remarks, we can conclude that the methods developed in this thesis fulfill the primary objective as described in Chapter 1. It is also clear that the methods, focused mainly on the observer design, are applicable to fault detection. Additionally, as a result of the research, all the methods reported in this thesis were published or submitted to conferences and journals. Future work will be done in order to improve the methods proposed in this thesis and address open issues. Some of these open problems are discussed below.

Extension of approaches 1 and 3 to the worst-case fault sensitivity

Three different approaches to design H_∞ fault detection observers were proposed in this document, as detailed in Chapter 3. The main idea of these approaches was to obtain a dynamic equation of the estimation error and then, the observer gains were synthesized by considering the $\mathscr{L}_2$-gain from the inputs to the residuals. As a result, different solutions were obtained in the LMI formulation.

- In particular the second approach, also called in the document the uncertain system approach, was extended to design a fault detection state observer with H_-/H_∞ performance as detailed in Chapter 4. However, the approaches 1 and 3 can be also extended to design observer with H_-/H_∞ performance. This will be explored in future works.

- From the author's point of view, finding an optimal solution to the H_-/H_∞ observer design is not completely explored in the literature, despite different algorithms were proposed in (Bouattour et al., 2011; Wang et al., 2007; Chadli et al., 2013a), in addition to the one proposed in Chapter 4. Therefore, designing an optimal numerical algorithm to solve the LMI system still remain an open and unsolved problem.

- Moreover, as given in Corollary 4.3.3, the solution to the H_- problem exists, if the term in the LMI 4.25, $D_f^T D_f - \beta^2 I > 0$. This condition is a necessary condition and is related to the mathematical criterion performance of H_- index. A solution to this problem could be found considering a limited frequency response $[0, \omega]$ instead of the infinite frequency response $[0, \infty)$.

Observer based controller

The main contribution of this thesis was to propose different methods to perform observer design for D-LPV systems with unmeasurable gain scheduling functions. Nevertheless, by considering the proposed methods, it could be possible to design observer based controllers to perform fault tracking and fault tolerant control. Note that the problem is not trivial because the problem involves the observer design and controllers with unmeasurable gain scheduling functions, which becomes a challenge issue. Therefore, the challenge is to design the fault diagnosis and control scheme by considering unmeasurable gain scheduling functions, which increase the complexity of the design due to the fact that the controller and the observer will be dependent on the estimation of the scheduling vector. This problem is not new, and some solutions have been proposed as detailed in (Nachidi and El Hajjaji, 2012; Moodi and Farrokhi, 2014), however the problem is not completely solved and represents a challenge and an important research issue.

The discrete-time case

As detailed in Chapter 2, most of the published methods for fault detection and isolation are dedicated to continuous-time systems and only a few works are related to discrete-time systems, e.g. in (Yoneyama, 2007) a H_∞ output feedback control was proposed by transforming the original system into an uncertain system; the uncertainties represent the differences between the estimated and the unmeasurable scheduling functions. A fault estimation observer poly-quadratically stable was given in (Astorga-Zaragoza et al., 2011). In (Boulkroune et al., 2013), it was proposed a fault detection observer based on mixed H_-/H_∞ index by considering global Lipschitz constraints and Lyapunov stability criterion. Recently, in (Chadli et al., 2013a), an H_-/H_∞ fault detection was proposed which considered the descriptor approach. Nevertheless, despite the reported works, we consider that the unmeasurable scheduling problem for discrete-time systems has not been fully addressed and remains an important open issue.

As remarked in this section, improving algorithms to find an optimal solution to the H_-/H_∞ problem, observer based controllers and discrete-time systems are just some of the open problems related to D-LPV systems. However, these topics are not unique and there are other open topics, as for example fault-tolerant controllers, delayed systems, networked systems, multiplicative and parametric faults, or the problem of guarantee observability for all convex combinations, which the authors want to address in future works. All these open topics illustrate the pertinence and importance of the study of descriptor systems with unmeasurable gain scheduling functions and its importance in future research.

Appendix A

Tutorial about descriptor linear time invariant systems

A.1 Introduction

A general representation of a linear time invariant system is given as

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{A.1}$$
$$y(t) = Cx(t) + Du(t)$$

for continuous-time, and

$$Ex(k+1) = Ax(k) + Bu(k) \tag{A.2}$$
$$y(k) = Cx(k) + Du(k) \tag{A.3}$$

for discrete-time. where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. E is singular and not necessarily diagonal, the behavior relationships are not necessarily dynamics.

The main difference between typical state-space systems and descriptors systems, is that the matrix E of (A.1)-(A.2) is singular and therefore not invertible.

Two of the most important properties of descriptor systems are solvability and regularity. If these properties hold, a singular system can be converted to an equivalent form which could be necessary to compute controllability, observability or stability. If the system is not regular, then the system can regularized. This properties are analyzed in the next sections.

A.2 Regularity

Property of regularity is related to the invertibility of matrix pencil pair (E, A). The following definitions hold for regularity.

Definition A.2.1. (Gantmacher, 1959) The system (A.1) is called regular if there exists a constant scalar γ such that $\det(\gamma E - A) \neq 0$ or equivalently, the polynomial $\det(sE - A)$ is not identically zero. In this case, we also say that the matrix pair (E, A), or the matrix pencil $sE - A$; is regular.

Equivalently, regularity can be defined as:

Definition A.2.2. The system (A.1), or the matrix pair (E, A), is called regular if there exist a pair of scalars such that:

$$\det(\alpha E - \beta A) \neq 0 \tag{A.4}$$

The following theorem gives some criteria for regularity:

Theorem A.2.3. *(Yip and Sincovec, 1981) Let matrices $(A, E) \in \mathbb{R}^{n \times n}$ the following statements are equivalents:*

1. (A, E) is regular
2. Let X_0= $(\ker(A))$, $X_i = \{x | Ax \in EX_{i-1}\}$then $\ker E \cap X_i = \{0\}$ for $i = 0, 1, 2, \ldots$
3. Let $Y_0 = \ker A^T$, $Y_i = \{x | Ax \in EY_{i-1}\}$then $\ker E \cap Y_t = \{0\}$ for $i = 0, 1, 2, \ldots$
4. Let
$$G(k) = \begin{pmatrix} E & & \\ A & E & \\ & A & \ddots \\ & \ddots & E \\ & & A \end{pmatrix} \in \mathbb{R}^{(k+1)n \times nk}$$
the $\operatorname{rank} G(k) = nk$ for $k = 1, 2, \ldots$
5. Let
$$F(k) = \begin{pmatrix} E & A & & & \\ & E & A & & \\ & & \ddots & \ddots & \\ & & & A & \\ & & & E & A \end{pmatrix} \in \mathbb{R}^{nk \times n(k+1)}$$
The $\operatorname{rank} F(k) = nk$ for $k = 1, 2, \ldots$

The conditions 2–5 in the above theorem are generally not convenient to verify. Moreover, for large k, the computation of $rank\, G(k)$ and $rank\, F(k)$ may subject to numerical problems. The following theorem gives a method for verifying the regularity of a given descriptor linear system based on its dynamics decomposition:

Theorem A.2.4. *(Duan, 2010) Let $A, E \in \mathbb{R}^{n\times n}$ with $rank\, E = n_0$, and $P, Q \in \mathbb{R}^{n\times n}$ be two nonsingular matrices satisfying*

$$QAP = \begin{bmatrix} E_0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A.5}$$

where $E_0 \in \mathbb{R}^{n_0\times n_0}$ is nonsingular matrices. Further let

$$QEP = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{A.6}$$

then the matrix (E, A) is regular if an only if

$$\det(A_{22} + A_{21}(\gamma E_0 - A_{11})^{-1} A_{12}) \neq 0 \tag{A.7}$$

holds for arbitrary choosing $\gamma \in \sigma(E_0, A_{11})$.

The above criteria is very convenient to use, but is very hard to implement computationally. Then the following theorem which is more easy from computational point of view:

Theorem A.2.5. *(Duan, 2010) The pair matrix (E,A), with $E,A \in \mathbb{R}^{n\times n}$ is regular if and only if*

$$max\{rank(\gamma_i E - A),\, i = 1,2,...,l+1\} = n \tag{A.8}$$

holds for an arbitrary selected set of distinct number $\gamma_i \in \mathbb{C}$, $i = 1,2,...l+1, l = deg\, det(sE - A)$

Example A.2.6. Consider the matrix pair (E, A) with:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

choosing $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = -1$ then is easy to verify:

$$rank(\gamma_1 E - A) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = 3 < n$$

$$rank(\gamma_2 E - A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = 4 = n$$

$$rank(\gamma_3 E - A) = \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = 4 = n$$

therefore according with the above theorem the system is regular.

A.3 Solvability

Solvability is defined as the existence of a unique solution for any given sufficiently differentiable $u(t)$ and any given admissible initial condition corresponding to the given $u(t)$.

Definition A.3.1. (Yip and Sincovec, 1981) (A, E) is solvable if the matrix pencil $E + \lambda A$ is regular. In other words the matricial relationship (E,A) is solvable if and only if $|sE - A| \neq 0$, or equivalently if there exist an scalar $\lambda \in \mathbb{C}$ such that $|\lambda E - A| \neq 0$.where $|\centerdot|$ is the determinant .

The following example illustrate the concept of solvability.

Example A.3.2. Marx et al. (2007a) Consider the RCL circuit given in Fig. A.1

Which is described for the following descriptor system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ \dot{i}_2(t) \\ \dot{i}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{C} & -R_2 & 0 \\ \frac{1}{C} & R_1 & R_2 \end{bmatrix} \begin{bmatrix} q(t) \\ i_2(t) \\ i_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ i_2(t) \\ i_3(t) \end{bmatrix}$$

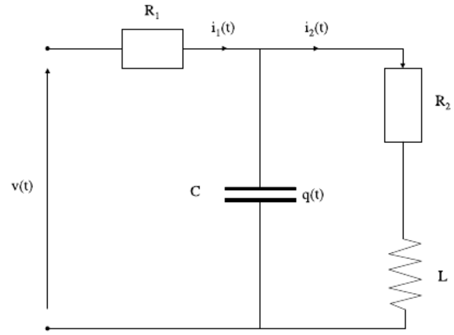


Figure A.1 – A electric circuit

By considering Theorem A.3.1, the solvavility of the descriptor model is verified as:

$$|\lambda E - A| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda L & 0 \\ & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{C} & -R_2 & 0 \\ \frac{1}{C} & R_1 & R_2 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda & 0 & -1 \\ -\frac{1}{C} & \lambda + R_2 & 0 \\ -\frac{1}{C} & -R_1 & -R_2 \end{bmatrix} \right|$$

$$|\lambda E - A| = \lambda(\lambda + R_2) - 0 - \left(\frac{R_1}{c} + \frac{\lambda + R_2}{C} \right)$$

Therefore due that $|\lambda E - A| \neq 0$, the system is solvable.

A.4 Stability

In typical state-space system we say that a system is stable if the eigenvalues of $(sI - A) \in \mathbb{C}^-$, for descriptor system the criterion is basically the same and is resumed in the next theorem:

Theorem A.4.1. *A regular descriptor system is stable if and only if*

$$\mathrm{eig}(E, A) = \subset \mathbb{C}^- = \{s | s \in \mathbb{C}, \mathrm{Re}(s) < 0\} \tag{A.9}$$

Clearly, the stability is defined by the position of the eigenvalues. If the system has negative real part then, is stable. If the system has zero real part then is critically stable and if has positive real part, then is unstable. Nevertheless, more general stability conditions can be achieved by considering the following Lyapunov method.

A.4.1 Stability via Lyapunov

Consider a regular descriptor system (E, A, C), the following hold:

O1 Stable if Theorem (A.4.1) hold

O2 impulse free if it exhibits no impulsive behavior;

O3 finite-dynamics detectable if $(O1)$ holds;

O4 finite-dynamics observable if $(O2)$ holds;

O5 impulse observable if $(O3)$ holds;

O6 S-observable if $(O2)$ and $(O3)$ hold;

O7 C -observable if $(O2)$ and $(O4)$ hold, where $(O1)$–$(O4)$ was given in the above section

According with this the principal approach of stability are based in the generalized Lyapunov equation. For state space system the Lyapunov equation is:

$$A^T X + XA + Y = 0,\ Y > 0 \tag{A.10}$$

For singular system the stability based in the generalized Lyapunov equation are showed in the next theorems:

Theorem A.4.2. *Ishihara and Terra (2002) Let (A, B) be regular and consider the following generalized Lyapunov equation (GLE)*

$$A^T XE + E^T XA + E^T YE = 0 \tag{A.11}$$

we have that:

a) if there exist matrices $X \geq 0$ and $Y \geq 0$ satisfying the GLE (A.11), the (E, A) is impulsive free and stable

b) if (E, A) is impulsive free and stable, for each $Y > 0$ there exist $X > 0$ solution of GLE (A.11). Furthermore, $E^T XE > 0$ is unique for each $Y > 0$

Example A.4.3. The following example illustrates the stability computation by mean of Lyapunov theory. Consider the following descriptor system

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} \quad (t) = \begin{bmatrix} 0 & 0 & 1 \\ 100 & -50 & 0 \\ 100 & 20 & 20 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)
$$

$$
y = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t)
$$

Them, it is easy to verify that there exist two matrices X and Y positive definite

$$
X = \begin{bmatrix} 0.072 & -0.001 & -0.004 \\ -0.001 & 0.001 & 0.000 \\ -0.004 & 0.000 & 0.000 \end{bmatrix} \; Y = \begin{bmatrix} 1 & -5. \times 10^{-08} & 0 \\ -5. \times 10^{-08} & 1 & 0 \\ 0 & 0 & 1 \times 10^{+06} \end{bmatrix}
$$

such that:

$$
A^T XE + E^T XA + E^T YE = 0
$$

whose eigenvalues are:

$$
\Lambda(X) = \begin{bmatrix} 0 \\ 0.0005 \\ 0.0725 \end{bmatrix}, \; \Lambda(Y) = \begin{bmatrix} 1 \\ 1 \\ 1. \times 10^{+06} \end{bmatrix}
$$

therefore the system is stable.

A.5 Equivalents Forms

Equivalence is a natural concept in the study of descriptor systems. The equivalence relation between two descriptor linear systems is determined by a transformation, which involves two nonsingular matrices. As also discussed in Chapter 2, an equivalent system can be used to separate the dynamic and the static part which can facility the study of asses properties. As also given in Chapter 2, the main most important equivalent transformation is the so called Kronecker-Weierstrass canonical form. However, other transformation that can be very useful as discussed below.

A.5.1 QR transformation

This equivalent transformation is based on the QR decomposition method that considers that any real square matrix A may be decomposed as

$$A = QR$$

where Q is an orthogonal matrix and R is an upper triangular matrix. More general, for the case of descriptor systems,it can be verified that there exist a non-singular P matrix such that:

$$PE = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix} \tag{A.12}$$

$$PA = \begin{bmatrix} \bar{A} \\ \bar{A}_1 \end{bmatrix} \tag{A.13}$$

$$PB = \begin{bmatrix} \bar{B} \\ \bar{B}_1 \end{bmatrix} \tag{A.14}$$

such that the descriptor system is restricted system equivalent to

$$\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) \tag{A.15}$$

$$\bar{y}(t) = \bar{C}x(t) \tag{A.16}$$

where

$$\bar{y}(t) = \begin{bmatrix} -\bar{B}_1 u(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^q \tag{A.17}$$

and

$$\bar{C} = \begin{bmatrix} \bar{A}_1 \\ C \end{bmatrix} \in \mathbb{R}^{q\times n} \tag{A.18}$$

with $q = m + p - r$

This equivalent transformation has prove to be particular useful in the design of state observer (see for instance Darouach and Boutayeb (1995); Darouach et al. (1996); Marx et al. (2007a); Hamdi et al. (2009)).

A.5.2 The inverse Equivalent Form

Theorem A.5.1. *(Duan, 2010) Consider the descriptor linear system* (E, A, B, C)*, with* $E, A \in \mathbb{R}^{n\times n}$*,*

$B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$ and (E,A) regular. Let γ be a scalar satisfying $\det(\gamma E - A) \neq 0$, *and define*

$$Q = (\gamma E - A)^{-1} \tag{A.19}$$

then:

$$(E,A,B,C) \overset{(I,Q)}{\Longleftrightarrow} (\hat{E},\hat{A},\hat{B},C)$$

with:

$$\begin{aligned}
\hat{E} &= QE = (\gamma E - A)^{-1} E \\
\hat{A} &= QA = (\gamma E - A)^{-1} A \\
\hat{B} &= QB = (\gamma E - A)^{-1} B
\end{aligned}$$

Furthermore the following relation hold:

$$\hat{A} = \gamma E - I$$

The system (A.1) is regular and, have the following restricted equivalent form:

$$\hat{E} = (\gamma E - I)x + \hat{B}u \tag{A.20}$$

$$y = Cx \tag{A.21}$$

A.6 Time Response

Let us consider the regular descriptor system (A.1), which can be decompose between the fast- (2.29) ant the slow-subsystem (2.28). Note that the slow-subsystem is an ordinary differential equation, then, the *slow sub-system* has a unique solution for any $x_1(0)$ for any piecewise continuous function $u(t)$, as Dai (1989)

$$x_1(t) = e^{A_1 t} x_1(0) + \int_0^t e^{A_1(t-\tau)} B_1 u(\tau) d\tau \tag{A.22}$$

On the other hand, if the input $u(t)$ is r_ι times piecewise continuously differentiable, r_ι is the nilpotent index as defined in (2.27). By considering continuously derivatives on both sides of (2.29), and left multiplying both sided by the nilpotent matrix N from (2.26), the following is obtained

$$\mathcal{N}^2\dot{x}_2 = Nx_2(t) + NB_2u(t)$$
$$\ldots$$
$$\mathcal{N}^{r_1}\dot{x}_2 = N^{r_1-1}x_2^{(r_1-1)}(t) + N^{r_1-1}B_2u^{(r_1-1)}(t)$$

where $x_2^{(i)}$ stands for the ith derivative of $x_2(t)$. Then, by considering the sum of all derivatives(also considering that $N^{r_1} = 0$), the response of the fast-subsystem due to the input $u(t)$ is given by

$$x_2(t) = -\sum_{i=0}^{r_1-1} N^i B_2 u^{(i)}(t) \tag{A.23}$$

Combining, slow subsystem (A.22) and fast subsystem (A.23), the *state response* of the system (A.1) is given by

$$x(t) = Q\begin{bmatrix} I \\ 0 \end{bmatrix} x_1(t) + Q\begin{bmatrix} I \\ 0 \end{bmatrix} x_2(t) \tag{A.24}$$

$$x(t) = Q\begin{bmatrix} I \\ 0 \end{bmatrix}\left(e^{A_1t}x_1(0) + \int_0^t e^{A_1(t-\tau)}B_1u(\tau)d\tau\right) + Q\begin{bmatrix} I \\ 0 \end{bmatrix}\left(-\sum_{i=0}^{r_1-1} N^i B_2 u^{(i)}(t)\right) \tag{A.25}$$

As discussed in Chapter 2, the time response of a descriptor system involves **(ith)** derivate of the input $u(t)$. This constraint must be taken in account, for example in the design of state observer.

Note that the time-repose of (A.1) depends on the matrix A_1, the initial condition $x_1(0)$, and the input $u(t)$ for $[0t]$, while $x_2(t)$ is a linear combination of derivatives of $u(t)$ at time t. In particular for $t > 0$, $t \to 0^+$, the following condition initial is imposed

$$x_1(0^+) = P\begin{bmatrix} I \\ 0 \end{bmatrix}\left(x_1(0) - \sum_{i=0}^{r_1-1} N^i B_2 u^{(i)}(0^+)\right)$$

which is the so-called consistent initial conditions imposed on initial value $x(0)$. Note that comparing with regular LTI systems. A LTI system has always one unique solution if the input is piecewise continuous. However, a descriptor system has a unique solution only for the consistent initial vector $x(0)$, and for the i times piecewise continuously differentiable input $u(t)$.

A.7 Controllability and Observability using the Laurent Expansion approach

A.7.1 Controllability

The controllability and observability of a descriptor system should be analyzed by the Laurent expansion approach given by (A.47). This approach was analyzed in (Koumboulis and Mertzios, 1999). This section is mainly based in the Koumboulis paper.

C controllability

Definition A.7.1. The system (A.1) is controllable if for every $x(0-)$ and every $\xi \in \mathbb{R}^n$ there exist a finite interval, $[0, T](T \geq 0)$, and a input function $u(t)\,(t \in [0, T])$, such that $x(T) = \xi$

The necessary and sufficient condition of controlability is established in the following theorem:

Definition A.7.2. The system (A.1) is controllable if and only if

$$\mathsf{rank}\left[\begin{array}{cccccc} \Phi_{-h}B & \dots & \Phi_{-1}B & \Phi_0 B & \dots & \Phi_{n-1}B \end{array}\right] = n \tag{A.26}$$

R. Controllability

Definition A.7.3. The system (A.1) is R-Controllable if for every $x(0-)$ there exist a finite time interval, $[0,T](T > 0)$, and a input function, $u(t)(t \in [0,T])$, such that $x(T) = 0$

Theorem A.7.4. *The system (A.1) is R-Controllable if and only if*

$$\mathsf{rank}[\Phi_0 B \cdots \Phi_{n-1}B] = \mathsf{rank}(\Phi_0) \tag{A.27}$$

Impulse Controllability

Definition A.7.5. The system is I-Controllable if if for every $x(0-)$ there exist a bounded input $u(t)(t \geq 0)$ such that $x(t)$ is finite $\forall t \geq 0$

Theorem A.7.6. *The system (A.1) is I-Controllable if and only if the following condition is satisfied*

$$\mathsf{rank}\left[\begin{array}{ccc} \Phi_{-h}B & \cdots & \Phi_{-2}B \end{array}\right] = \mathsf{rank}(\Phi_{-2}) \tag{A.28}$$

A.7.2 Observability

C-Observability

Theorem A.7.7. *The system (A.1) is C-Observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C\Phi_{-h} \\ \vdots \\ C\Phi_{-1} \\ C\Phi_0 \\ \vdots \\ C\Phi_{n-1} \end{bmatrix} = n \tag{A.29}$$

R-Observability

Theorem A.7.8. *The system (A.1) is R-Observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C\Phi_0 \\ \vdots \\ C\Phi_{n-1} \end{bmatrix} = \operatorname{rank}(\Phi_0) \tag{A.30}$$

I-Observability

Theorem A.7.9. *The system (A.1) is I-Observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C\Phi_{-h} \\ \vdots \\ C\Phi_{-2} \end{bmatrix} = \operatorname{rank}(\Phi_{-2}) \tag{A.31}$$

A.8 Controllability and Observability using the Kronocker Weisstrauss r.s.e. approach

In descriptor systems there exists principally three types of controllability/observability: Complete, Reachable and Impulsive. For the case of C-controllability, it is related with the fact that the system can be controlled with any initial conditions Duan (2010). R-controllability means that the system only can be controlled by a reduced set of initial conditions and I-Controllability means that impulsive signals can be controlled.

The controllability in descriptor systems can be computed in different ways, e.g. using Laurent expansion, equivalent systems or from direct approaches. In the Laurent expansion approach, the controllability is related with the dimension of (E,A) and the rank of the Laurent expansion matrices. In the restricted system approach the controllability can be found by the analysis of controllability of slow (2.28) and fast (2.29) subsystems (Duan, 2010). While in the direct form approach the controllability is found by analyzing matrices (E,A,B) (see (Yip and Sincovec, 1981)). Similar definitions are considered for the case of observability. The following lemmas details these methods.

A.8.1 Controllability

C-controllability

Consider the descriptor system whose corresponding pencil $sE-A$ is of index v and the slow and fast subsystems are given by (2.28) and (2.29), respectively.

Definition A.8.1. A reachable set $R(x0)$ of system (A.1) from an initial condition $x(0)=x0$ is given by:

$$R(x_0) = \omega \in \mathbb{R}^n : \exists u \in C^v \exists t_1 > 0 \text{ such that } x(t_1) = \omega. \tag{A.32}$$

The reachable set R is defined as the union of $R(x_0)$ over all consistent $x_0 \in \mathbb{R}^n$.

System (A.1) is called (Sokolov, 2006):

i) completely observable (*C-Observable*), if for any $\omega \in \mathbb{R}^n$, $x(0) \in \mathbb{R}^n$ and $t_1 > 0$ exist a control $u \in \mathbb{C}^v$ such that $x(t_1) = \omega$.

ii) *R-Controllable, if for any* $\omega \in \mathbb{R}$, *any consistent initial value* $x(0)$, *and* $t_1 > 0$ *there exist a control* $u \in \mathbb{C}^v$ such that $x(t_1) = \omega$.

iii) Impulse Controllable (Imp-Controllable), if there exist a feedback matrix F such that the state response of the close loop system $[E,A+BF,B,C,D]$ does not have impulsive components for all initial initial conditions and for all inputs $u(t) \in \mathbb{C}^v$.

iv) Strongly Controllable (S-Controllable), if is the both *R-Controllable* and *R-Controllable.*

Theorem A.8.2. *Consider the system (A.1) with fast subsystem (2.29) and slow subsystem (2.28):*

The slow subsystem (2.28) is C-Controllable if and only if

$$\text{rank} \left[sI - A_1 B_1 \right] = n, \ \ \forall n \in \mathbb{C}\, s\, \text{finite}$$

The fast subsystem (2.29) is C-Controllable if and only if:

$$\text{rank}\left(\begin{bmatrix} N & B_2 \end{bmatrix}\right) = n_2$$

The system (A.1) is C-Controllable if and only if both slow (2.28) and fast (2.29) subsystem are C-Controllable

R-Controllability

Theorem A.8.3. *Consider the system (A.1) with fast subsystem (2.29) and slow subsystem (2.28)is R-Controllable if and only if the slow subsystem (2.28) is C-Controllable.*

I-Controllability

Theorem A.8.4. *Consider the system (A.1) with fast subsystem (2.29) :*

System (A.1) is I-Controllable if and only if its fast subsystem (2.29) is I-Controllable

The fast subsystem 2.29 is I-Controllable if and only if one of the following conditions holds:

Theorem A.8.5. $\text{Image}\, N = \text{Image}\begin{bmatrix} NB_2 & N^2B_2 & \dots & N^{h-1}B_2 \end{bmatrix}$

$\ker N + \text{Image}\, Q_C[N, B_2] = \mathbb{R}^{n_2}$

$\text{Image}\, N + \ker N + \text{Image}\, B_2 = \mathbb{R}^{n_2}$

A.8.2 Observability

C Observability

Theorem A.8.6. *Consider the descriptor system (A.1) with fast subsystem (2.29) and slow subsystem (2.28):*

The slow subsystem (2.28) is C-Observable if and only if:

$$\text{rank}\begin{bmatrix} sI - A_1 \\ C_1 \end{bmatrix} = n_1,\ \forall s \in \mathbb{C},\ s\,\text{finite} \tag{A.33}$$

The fast subsystem (2.29) is C-Observable if and only if:

$$\text{rank}\begin{bmatrix} N \\ C_2 \end{bmatrix} = n_2$$

The system (A.1) is C-Observable if and only if both slow (2.28) and fast (2.29) subsystem are C-Observable.

R Observability

Theorem A.8.7. *(Duan, 2010) The regular descriptor linear system (A.1) with slow subsystem (2.28) and fast subsystem (2.29) is R-observable if and only if its slow subsystem (2.28) is observable or equivalent if the condition (A.33) holds.*

Imp. Observability

The I-observability of system (A.1) is concerned with observing the impulse terms in the system state response from the output data of the system.

Theorem A.8.8. *(Duan (2010))Let (2.28) and (2.29) be the slow subsystem and the fast subsystem of the regular descriptor linear system (A.1), respectively.*

System (A.1) is I-observable if and only if its fast subsystem (2.29) is I-observable.

The fast subsystem (2.29) is I-observable if and only if one of the following conditions holds:

Theorem A.8.9. $\ker\left(Q_o\left[N, C_2\right]N\right) = \ker N$

$\ker\left(Q_o\left[N, C_2\right] \cap \mathrm{Image}\, N = \{0\}\right.$

$\ker N \cap \mathrm{Image}\, N \cap \ker C_2\, \{0\}$

Note that:

$$Q_o\left[N, C_2\right]N = \begin{bmatrix} C_2N \\ C_2N^2 \\ \vdots \\ C_2N^{h-1} \\ 0 \end{bmatrix} \tag{A.34}$$

then we have:

$$\ker\left(Q_o\left[N, C_2\right]N\right) = \ker \begin{bmatrix} C_2N \\ C_2N^2 \\ \vdots \\ C_2N^{h-1} \end{bmatrix} \tag{A.35}$$

Impulse observability guarantees the ability to uniquely determine the impulse behavior in $x(t)$ from information of the impulse behavior in output. C-observability and R-observability are on the finite value terms in the state response, while impulse observability focuses on the impulse terms that take infinite values.

A.9 Direct criteria for Controllability and Observability

In the previous sections have used two types approach for determine the controllability and observability of a descriptor system in the form (A.1). The first type uses the approach of the Laurent expansion and the second is based on the decomposition of the system as slow and fast subsystem. Both are very important approaches because in some problems, as in the case of the discretization, one must work with the expansion of Laurent and others in the Kronecker canonical form. However, when the system is regular, you can work with a direct approach to determining the different types of controllability and observability, this concerted approach are summarized in the following theorems.

A.9.1 C-Controllability and C-Observability

Theorem A.9.1. *(Duan, 2010)Consider the regular descriptor linear system (A.1) with its slow subsystem (2.28) and fast subsystem (2.29).*

The slow subsystem (2.28) is C-controllable if and only if

$$\operatorname{rank}\begin{bmatrix} sE-A & B \end{bmatrix} = n,\ \forall s \in \mathbb{C},\ s\,\text{finite} \tag{A.36}$$

The fast subsystem (2.29) is C-Controllable if and only if

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = n \tag{A.37}$$

The system (A.1) is C-Controllable if and only if (A.36) and (A.37) hold, or

$$\operatorname{rank}\begin{bmatrix} \alpha E-\beta A & B \end{bmatrix} = n,\ \forall(\alpha,\beta) \in \mathbb{C}^2\backslash\{0,0\} \tag{A.38}$$

(Duan, 2010) Consider the regular descriptor linear system (A.1) with its slow subsystem (2.28) and fast subsystem (2.29).

The slow subsystem (2.28) is C-observable if and only if

$$\operatorname{rank}\begin{bmatrix} sE-A \\ C \end{bmatrix} = n,\ \forall s\in\mathbb{C},\ s\,\text{finite} \tag{A.39}$$

The fast subsystem (2.29) is C-observable if and only if

$$\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n \tag{A.40}$$

The system (A.1) is C-Observable if and only if (A.39) and (A.40) hold, or

$$\operatorname{rank}\begin{bmatrix} \alpha E-\beta A \\ C \end{bmatrix} = n,\ \ \forall(\alpha,\beta)\in\mathbb{C}^2\backslash\{0,0\} \tag{A.41}$$

A.9.2 R-Controllability and R-Observability

Theorem A.9.2. *(Duan, 2010) The regular descriptor system (A.1) is R-Controllable if and only if*

$$\operatorname{rank}\begin{bmatrix} sE-A & B \end{bmatrix} = n,\ \forall s\in\mathbb{C},\ s\,\text{finite} \tag{A.42}$$

(Duan, 2010) The regular descriptor system (A.1) is R-Observable if and only if

$$\operatorname{rank}\begin{bmatrix} sE-A \\ C \end{bmatrix} = n,\ \forall s\in\mathbb{C},\ s\,\text{finite} \tag{A.43}$$

A.9.3 I-Controllability and I-Observability

Theorem A.9.3. *(Duan, 2010) The regular descriptor system (A.1) is I-Controllable if and only if*

$$\operatorname{rank}\begin{bmatrix} E & AV_\infty & B \end{bmatrix} = n \tag{A.44}$$

holds for an arbitrary matrix $V_\infty \in \mathbb{R}^{n\times(n-n_0)}$ whose columns span $\ker E$ or equivalently

$$\operatorname{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n+\operatorname{rank}E \tag{A.45}$$

(Duan, 2010) The regular descriptor system (A.1) is I-Observable if and only if

$$\operatorname{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \operatorname{rank} E \tag{A.46}$$

A.10 Discretizing Continuous Descriptor Systems

The synthesis presented in this section can be consulted on details in (Karampetakis, 2003).

Considers a regular descriptor system in the form of (A.1). Its resolvent matrix can be expressed in a power series of the Laurent Expansion of $(sE-A)^{-1}$, as

$$\Phi(s) = (sE-A)^{-1} = \Phi_{-k}s^{k-1} + \cdots + \Phi_{-2}s + \Phi_0 s^0 + \Phi_1 s^{-1} + \ldots \Phi_k s^{-k-1} \tag{A.47}$$

$$= \sum_{k=h}^{\infty} \Phi_k (E,A)\, s^{-k-1} \tag{A.48}$$

whit $k = 1,2,...h$, where h is the nilpotent index of $(sE-A)$, and Φ are the fundamental matrix defined in Lewis (1985, 1990). The following properties of the fundamental matrix are well know:

Theorem A.10.1. *If the pair (E,A) is regular and Φ defined by (A.47), then:*

$\Phi E - \Phi_{k-1}A = I\delta_k$, *where* δ*is the Kronecker delta*

$E\Phi_k - A\Phi_{k-1} = I\delta$

$\Phi_k = (\Phi_0 A)^k \Phi_0 = \Phi_{k-1}A\Phi_0 \quad k = 1,2,..h-1$

$\Phi_k = (-\Phi_{-1}E)^{-k-1}\Phi_{-1}, \;\; k < 0$

or equivalent by relations (Karampetakis, 2003; Karampetakis and Vologiannidis, 2004):

$$\Phi_k E = \Phi_{k-1}A,\; k = -h,..,-2,-1,\, with\, \Phi_{-h-1} = 0 \tag{A.49}$$

$$\Phi_0 E - \Phi_{-1}A = I \tag{A.50}$$

$$\Phi_k = \Phi_0 (A\Phi_0)^k = \Phi_0 A \Phi_{k-1},\, k = 1,2,... \tag{A.51}$$

$$\Phi_{-k} = -\Phi_{-1}E\Phi_{-k+1} = \Phi_{-1}(-E\Phi_{-1})^k, k = 2,3,...,h \tag{A.52}$$

based on this theorem, the following properties are derived

$$\Phi_i E \Phi_j = \Phi_j E \Phi_i, \ (\forall i,\ j) \tag{A.53}$$

$$\Phi_i E \Phi_j = \begin{cases} -\Phi_{i+j} & i<0,\ j>0 \\ \Phi_{i+j} & i \geqslant 0,\ j \geqslant 0 \\ 0 & ij \leqslant 0 \, and \, |i|+|j| \neq 0 \end{cases} \tag{A.54}$$

$$\Phi_i A \Phi_j = \begin{cases} -\Phi_{i+j+1} & i<0,\ j>0 \\ \Phi_{i+j+1} & i \geqslant 0,\ j \geqslant 0 \\ 0 & ij \leqslant 0 \, and \, |i|+|j| \neq 0 \end{cases} \tag{A.55}$$

The fundamental matrices Φ_0 and Φ_{-1} are obtained by the Drazin inverse of the matrix A (Bernstein, 2009; Stykel, 2006; Jun, 1994, 2002):

$$A^D = S \begin{bmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \tag{A.56}$$

This should modified using the the transformations matrix given by Theorem 2.4.3, P and Q of KW r.s.e: (Stykel, 2006)

$$\Phi_k = P \begin{bmatrix} A_1^k & 0 \\ 0 & 0 \end{bmatrix} Q \ k \geq 0 \tag{A.57}$$

$$\Phi_{-1} = P \begin{bmatrix} 0 & 0 \\ 0 & -N^{k-1} \end{bmatrix} Q \ k > 0 \tag{A.58}$$

then:

$$\Phi_0 = P \begin{bmatrix} A_1^0 & 0 \\ 0 & 0 \end{bmatrix} Q \tag{A.59}$$

$$\Phi_{-1} = P \begin{bmatrix} 0 & 0 \\ 0 & -N^0 \end{bmatrix} Q \tag{A.60}$$

Then the Laurent expansion of$(sE-A)^{-1}$ is:

$$(sE-A)^{-1} = \Phi_{-2}s^2 + \Phi_{-1}s + \Phi_0 + \Phi_1\frac{1}{s}$$

Then, the solutions of a D-LTI system in terms of the resolvent matrix is (Koumboulis and Mertzios, 1999)

$$\begin{aligned} x(t) &= e^{\Phi_0 At}\Phi_0 Ex(0-) + \int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 Bu(\tau)d\tau + \sum_{i=0}^{h-1}\Phi_{-i-1}Bu^{(i)}(t) \\ &+ \sum_{i=0}^{h-1}\Phi_{-i-1}\left(\delta^{(i)}(t)Ex(0-) + \sum_{j=0}^{i-1}\delta^{(i-j-1)}Bu^{(j)}(0-)\right) \end{aligned} \tag{A.61}$$

where Φ_n is obtained by the Laurent expansion of $(sE-A)^{-1}$.

State space discretization of homogeneous descriptor system

With $u = 0$, the descriptor system (A.1) is:

$$E\dot{x}(t) = Ax \tag{A.62}$$

the solution according to A.61 is:

$$x(t) = e^{\Phi_0 At}\Phi_0 Ex(0-) \tag{A.63}$$

based on this the following theorem is verifiable:

Theorem A.10.2. *(Karampetakis, 2003) The response of the singular system (A.62) at the sample times $kT = 0, 1, \ldots$ is given by the response of the discrete time system*

$$x((k+1)T) = \tilde{A}x(kT),\ x(0) = \Phi_0 Ex(0-) \tag{A.64}$$

where:

$$\tilde{A} = e^{\Phi_0 AT} \tag{A.65}$$

State space discretization of non-homogeneous singular systems

Consider the singular system (A.1) with the solution given by (A.61). The solution (A.61) may be rewritten as:

$$
\begin{aligned}
x(t) &= e^{\Phi_0 At}\Phi_0 Ex(0-)+\int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 Bu(\tau)d\tau+\sum_{i=0}^{h-1}(-\Phi_{-1}E)^i\Phi_{-1}Bu^{(i)}(t) \\
&\quad +\sum_{i=0}^{h-1}(-\Phi_{-1}E)^i\Phi_{-1}\left(\delta^{(i)}(t)Ex(0-)+\sum_{i=0}^{i-1}\delta^{(i-j-1)}Bu^{(j)}(0-)\right) \qquad \text{(A.66)}
\end{aligned}
$$

Then, the equivalent discrete system of (A.1) is given by the following Theorem.

Theorem A.10.3. *(Karampetakis, 2003) Using a zero-order hold of the input $u(t)$ and a first order hold of the derivatives of the input $u(t)$, the continuous time non-homogeneous singular system is discretized to yield the state space system:*

$$
x((k+1)T) = \tilde{A}x(kT)+\tilde{B}(\sigma)u(kT) \qquad \text{(A.67)}
$$

$$
x(0) = \Phi_0 Ex(0-)+\sum_{i=0}^{h-1}(-\Phi_{-1}E)^i\Phi_{-1}Bu^{(i)}(0-) \qquad \text{(A.68)}
$$

where:

$$
\tilde{A} = e^{\Phi_o AT} \qquad \text{(A.69)}
$$

$$
\hat{B}_0 = \left[\int_0^T e^{\Phi_0 A\omega}d\omega\right]\Phi_0 B+\sum_{i=1}^{h}(-1)^i\Phi_{-1}BT^{1-i} \qquad \text{(A.70)}
$$

$$
\hat{B}_1 = \sum_{j=l}^{h}(-1)^{j-l}\Phi_{-j}BT^{1-j}\begin{pmatrix} j \\ j-l \end{pmatrix} \; l=1,2,\ldots,h \qquad \text{(A.71)}
$$

$$
\hat{B}(\sigma) = \sum_{i=0}^{h}\hat{B}_i\sigma^i \; with \; \sigma u(kT)=u((k+1)T) \qquad \text{(A.72)}
$$

Singular system discretization of non-homogeneous singular systems

Consider the continuous singular system:

$$
\underbrace{\begin{bmatrix} \rho I_n-\Phi_0 A & 0 \\ 0 & I_n+\rho\Phi_{-1}E \end{bmatrix}}_{\rho\tilde{E}-A}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\tilde{x}(t)}=\underbrace{\begin{bmatrix} \Phi_0 B \\ \Phi_{-1}B \end{bmatrix}}_{\tilde{B}}u(t) \qquad \text{(A.73)}
$$

Theorem A.10.4. (Karampetakis, 2003) *There is a bijective map between the solution spaces—initial conditions of the systems (A.1) and (A.73).*

According with this theorem, instead of discretize system (A.1), we may discretize the system (A.73), giving rise to the following theorem:

Theorem A.10.5. *(Karampetakis, 2003) Using a zero-order hold approximation of the inputu(t) and first-order hold approximation of the derivatives of the inputu(t), the continuous time non-homogeneous singular system is discretized to yield the singular state space system:*

$$\left\{ \begin{array}{rcl} x_1((k+1)T) & = & \tilde{A}x_1(kT) + \tilde{B}_1 u(kT) \\ \tilde{E}_1 x_2((k+1)T) & = & x_2(kT) + \tilde{B}_2 u(kT) \end{array} \right\} \tag{A.74}$$

$$x(kT) = \left[\begin{array}{cc} I_n & I_n \end{array} \right] \left[\begin{array}{c} x_{-1}(kT) \\ x_2(kT) \end{array} \right]$$

where $\tilde{A}$ is given by (A.69), then:

$$\tilde{B}_1 = \left[\int_0^T e^{\Phi_0 A \tau} d\tau \right] \Phi_0 B \tag{A.75}$$

$$\tilde{E}_1 = (\Phi_{-1}E - T \times I_n)^{-1} \Phi_{-1} E \tag{A.76}$$

$$\tilde{B}_2 = T(\Phi_{-1}E - T \times I_n)^{-1} \Phi_{-1} B \tag{A.77}$$

The following algorithm is proposed to solve the Theorem.

Algorithme A.1 Given a continuous system in the form (A.1), find the corresponding discrete system in the form of (A.74).

Step 1 Check if the pair (E,A) is regular, if $\det(\gamma E - A) \neq 0$, then continue.

Step 2 Find the matrices (P, Q) of (2.4.3)

Step 3 Compute the Drazin inverse, then find Φ_0 and Φ_{-1}

Step 4 Compute $\Phi_{-k} = -\Phi_{-1}E\Phi_{-k+1} = \Phi_{-1}(-E\Phi_{-1})^k$, for $k = 2,3,\ldots,h$ and $\Phi_k = \Phi_0 (A\Phi_0)^k = \Phi_0 A \Phi_{k-1}$, $k = 1,2,\ldots$

Step 5 Compute $\tilde{A}$, $\tilde{B}_1$, $\tilde{E}$, $\tilde{B}_2$ from (A.69, A.75, A.76) and (A.77) respectively.

Step 6 Rewrite equations in the form of (A.74)

This algorithm was programed in the D-LPV toolbox, detailed in Chapter 2.

Appendix B

Linear Matrix Inequalities

The objective of this Appendix is to present a collection of useful properties and Lemmas related to linear matrix inequalities. All the lemmas are presented without proof, but they can be consulted in the given references.

B.1 LMIs properties

Given the following LMI

$$X \geq 0$$

the following properties are equivalent

1. $X + Y \geq Y$
2. $Y^T XY \geq 0$, Y is not-singular
3. $X - X \geq -X$ i.e $0 \geq X$

B.2 Important Lemmas

Lemma B.2.1. *(Boyd et al. (1994)) Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

where $Q = Q^T$, $R = R^T$, is equivalent to

$$\begin{aligned} R &> 0 \\ Q - SR^{-1}S^T &> 0 \end{aligned}$$

Lemma B.2.2. *(Boyd et al. (1994)) The Schur complement result of section can be generalized to non-strict inequalities as*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$$

where $Q = Q^T$, $R = R^T$,

$$\begin{aligned} R &> 0 \\ Q - SR^{\dagger}S^T &> 0 \end{aligned}$$

where $R^{\dagger}$ denotes the Moore-Penrose inverse of R.

Lemma B.2.3. *Kim et al. (1995) Suppose that $\beta(\in \mathbb{R}) > 0$, a matrix P is symmetric positive definite and $S = S^{-1}$. Then, for an arbitrary square matrix A, $\beta P - A^T PA > 0$ if and only if*

$$\begin{bmatrix} \beta S & A \\ A^T & S^{-|} \end{bmatrix} > 0$$

Lemma B.2.4. *Kim et al. (1995) If $FF^T \leq I$, it holds that*

$$\begin{bmatrix} DD^T & 0 \\ 0 & E^T E \end{bmatrix} + \begin{bmatrix} 0 & DFE \\ E^T F^T D^T & 0 \end{bmatrix} \geq 0$$

where $D \in \mathbb{R}^{n\times s}$, $E \in \mathbb{R}^{q\times n}$, and $F \in \mathbb{R}^{s\times q}$.

Lemma B.2.5. *Boyd et al. (1994) The matrix inequality*

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \geq 0 \tag{B.1}$$

holds if and only if

$$
\begin{aligned}
Z_3 &\geq 0 \\
Z_1 - Z_2 Z_3^{\dagger} Z_2^T &\geq 0 \\
Z_2(I - Z_3 Z_3^{\dagger}) &= 0
\end{aligned}
$$

Lemma B.2.6. *Mansouri et al. (2008) For real matrices A, B, W, Y, Z and a regular matrix Q with appropiate dimensions one has*

$$
\begin{bmatrix} Y + B^T Q^{-1} B & W^T \\ W & Z + AQA^T \end{bmatrix} \leq 0 \Rightarrow \begin{bmatrix} Y & W^T + B^T A^T \\ W + AB & Z \end{bmatrix} \leq 0
$$

Lemma B.2.7. *Garcia and Bernussou (1995) For the vectors $x, y \in \mathbb{R}$ and an arbitrary matrix $D \in \mathbb{R}^{n \times s}$ and $E \in \mathbb{R}^{q \times n}$*

$$
\max \left\{ \left(x^T D F E y\right)^2 : F^T F < I,\, F \in \mathbb{R}^{s \times q} \right\} = x^T D D^T x y^T E^T E y
$$

Lemma B.2.8. *(Schur complement of a negative definite matrix) (Yang et al. (2005)) Given constant matrices Q, R, and S where $Q = Q^T$ and $0 < R = R^T$. Then $Q + S^T R^{-1} S < 0$ if and only if*

$$
\begin{bmatrix} Q & S^T \\ S & -R \end{bmatrix} < 0
$$

or equivalently

$$
\begin{bmatrix} -R & S \\ S^T & Q \end{bmatrix} < 0
$$

Lemma B.2.9. *(Yang et al. (2005)) Let M, H,and E be real matrices of appropriate dimensions, with Γ satisfying $\Gamma^T \Gamma \leq I$, then*

$$
M + H \Gamma E + E^T \Gamma^T H^T < 0
$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$
M + \varepsilon E^T E + \frac{1}{\varepsilon} H H^T < 0
$$

or equivalently

$$
\begin{bmatrix} M & H & \varepsilon E^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon E & 0 & -\varepsilon I \end{bmatrix} < 0
$$

Lemma B.2.10. *(Wang and Yang (2013)) The following conditions are equivalent:*

1. There exist a symmetric matrix $P > 0$ such that

$$A^T PA + Q < 0$$

2. There exist a symmetric matrix $P > 0$ and a matrix G such that

$$\begin{bmatrix} Q & -A^T G^T \\ -GA & P - G - G^T \end{bmatrix} < 0$$

Lemma B.2.11. *(Khargonekar et al. (1990)) For any matrices X and Y with appropriate dimensions, the following is verified*

$$X^T Y + Y^T X \leq X^T \beta X + Y^T \frac{1}{\beta} Y$$

for any $\beta > 0$.

Lemma B.2.12. *Consider two real matrices X, Y, and $F(t)$ with appropriate dimensions, for any positive scalar β, the following inequality is verified*

$$X^T F Y + Y^T F^T X \leq \beta X^T X + \beta Y^{-1T} Y > 0$$

Lemma B.2.13. *Khosrowjerdi and Barzegary (2013)For any $x, y \in \mathbb{R}^n$ and any positive definite matrix $P \in \mathbb{R}^{n \times n}$, we have*

$$2x^T y \leq x^T P x + y^T P^{-1} y$$

Lemma B.2.14. *Ibrir (2004)For any given vectors α, β and positive definite matrix P with compatible dimension, one has*

$$\alpha^T \beta + \beta^T \alpha \leq \alpha^T P \alpha + 2\beta^T P^{-1} \beta$$

Lemma B.2.15. *Gao and Ding (2007) All $Z \in \mathbb{R}^{n \times n}$ satisfying*

$$Z^T E = EZ \geq 0$$

can be parametrized as

$$Z = WE^T + E_r^{\perp} Q$$

where $W \geq 0 \in \mathbb{R}^{n \times n}$and $Q \in \mathbb{R}^{(n-r) \times n}$are parameter matrices, $E^{\perp} \in \mathbb{R}^{r \times (n-r)}$is a matrix such that $EE^{\perp} = 0$ and $\mathsf{rank}(E_r^{\perp}) = n - \mathsf{rank}(E) = n - r$. Furthermore, when Z is nonsingular, $W > 0$.

Lemma B.2.16. *Xu and Lam (2006) All $Z \in \mathbb{R}^{n\times n}$ satisfying*

$$E^T Z = Z^T E \geq 0$$

can be parametrized as

$$Z = PE + SX$$

where $P \geq 0 \in \mathbb{R}^{n\times n}$and $X \in \mathbb{R}^{(n-r)\times n}$are parameter matrices, $S \in \mathbb{R}^{n\times(n-r)}$is any matrix with full column rank and satisfies $E^T S = 0$.

Lemma B.2.17. *Tuan et al. (2001) Consider*

$$\sum_{i=1}^{h}\sum_{j=1}^{h} \rho_i(x(t))\rho_j(x(t))\mathcal{M}_{ij} \leq 0 \tag{B.2}$$

where $\rho_i(x(t))$ satisfies convexity. Then (B.2) is fulfilled if the following condition hold

$$\begin{aligned} \mathcal{M}_{ij} &< 0, i \in [1,2,...,h], \\ \frac{2}{h-1}\mathcal{M}_{ii} + \mathcal{M}_{ij} + \mathcal{M}_{ji} &\leq 0, 1 \leq i \neq j \leq h \end{aligned}$$

Definition B.2.18. Given a scalar $\gamma > 0$, the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ y(k) &= Cx(k) + Dw(k) \end{aligned} \tag{B.3}$$

is said to be stable with H_∞ disturbance attenuation γ if it is exponentially stable and input-output stable with $\| y \|_{L_2}^2 \leq d^2 \| w \|_{L_2}^2$ for some $0 < d < \gamma$.

Let $\gamma > 0$ be given, the system (B.3) is stable with H_∞ disturbance attenuation γ if and only if there exist a matrix $X > 0$ such that

$$\begin{bmatrix} -X & 0 & A^T & C^T \\ \star & -\gamma^2 I & B^T & D^T \\ \star & \star & -X^{-1} & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0 \tag{B.4}$$

Lemma B.2.19. *Daafouz and Bernussou (2001) The following conditions are equivalent:*

There exist a symmetric matrices $P_i > 0, P_j > 0$ such that

$$A_i^T P_j A_i - P_i > 0 \tag{B.5}$$

is equivalent to

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} \geq 0 \tag{B.6}$$

if and only if there exist symmetric positive definite matrices S_i, S_j and matrices $G_i$of appropriate dimensions for all $i, j \in [1, 2, ..., h]$. With $P(\zeta_k) = \sum_{i=1}^{h} \zeta_i(k) S_i^{-1}$.

Lemma B.2.20. *(Finsler's Lemma, Skelton et al. (1998); Asemani and Majd (2012)) let $\zeta \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{s\times n}$, $\mathsf{rank}(B) < n$, the*

$$\zeta^T Q \zeta < 0, \forall \zeta \neq 0, \textit{such that} B\zeta = 0$$

holds if and only if any of the following conditions holds:

1. $B_\perp^T Q B_\perp < 0$, where $B_\perp$ is such that $BB_\perp = 0$ and $BB^T + B_\perp^T B_\perp > 0$;
2. $\exists \mu \in \mathbb{R} : Q - \mu B^T B < 0$;
3. $\exists V \in \mathbb{R}^{n\times s} : Q + VB + B^T V^T < 0$

Lemma B.2.21. *Chadli et al. (2013a) Relaxation: For a LMI of the form*

$$\Gamma_{\zeta(t)\zeta(t)} = \sum_{i=1}^{h} \sum_{j=1}^{h} \rho_i(\zeta(t)) \rho_j(\zeta(t)) \Gamma_{ij} < 0 \tag{B.7}$$

where matrices Γ_{ij} are linearly independent on the unknown variables $\zeta(t)$. Condition (B.7) is fulfilled provided the following conditions hold $\forall i, j \in [1, 2...h]$ and $i \neq j$

$$\Gamma_{ij} < 0 \tag{B.8}$$

$$\frac{2}{h-1} \Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} \leq 0 \tag{B.9}$$

Appendix C

Technical results in tracking controllers and discrete-time systems

This Appendix presents some results that are not included in the main document due to its pertinence with the main objectives. These results include a fault detection and tracking controller for LPV systems, and a fault detection scheme based on observers for discrete-time D-LPV systems.

C.1 Fault detection and tracking controller

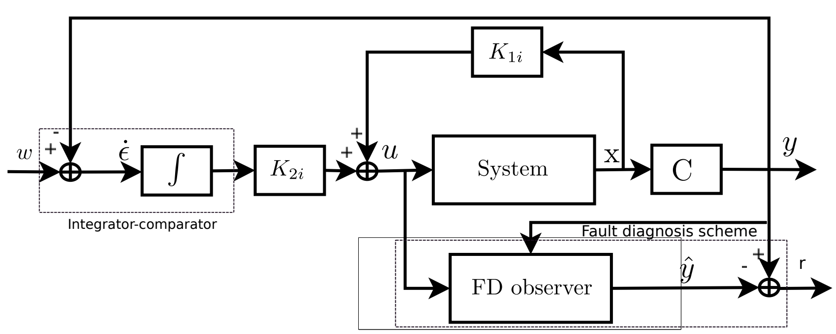


Figure C.1 – Fault diagnosis and Tracking controller scheme

Consider the control scheme displayed in Fig. C.1 which represents a fault diagnosis and tracking controller system. The system is modeled as LPV system given by

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{h} \rho_i(x(t)) \left[A_i x(t) + B_i u(t) + R_i d(t)\right] \\ y(t) &= Cx(t) \end{aligned} \tag{C.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^q$, and $y(t) \in \mathbb{R}^p$ are the state vector, the control input, the disturbances, and the measured vector respectively. A_i, B_i, R_i and, C and are constant matrices of appropriate dimensions. $\rho_i(x(t))$ are scheduling functions which depend on $x(t)$.

By assuming observable outputs and to generate the residuals, a robust fault diagnosis observer is described by the following equations is considered

$$\dot{z}(t) = \sum_{i=1}^{h} \rho_i(x(t)) \left[N_i z(t) + G_i u(t) + L_i y(t)\right] \tag{C.2}$$

$$\hat{x}(t) = z(t) + T_2 y(t)$$

$$r(t) = W(y(t) - C\hat{x}(t)), \tag{C.3}$$

where $z(t)$ represents the state vector of the observer, $\hat{x}(t)$ the estimated state vector. N_i, G_i, L_i, and T_2 are the gain matrices of (C.2) to be synthesized. $r(t)$ is the residual signal and W the residual weighting matrix to determine. The gain matrices of the fault diagnosis observer (C.2) are designed to guarantee the convergence of the state estimation error and maximize the robustness against disturbances $d(t)$.

The second objective is to design a feedback controller such that the steady-state response tends to $\lim_{t\to\infty} y(t) := w(t)$, where $w(t)$ is the desired position. To reach the desired position an integrator comparator is added as shown in Fig. C.1, such that

$$\dot{\varepsilon}(t) = w(t) - y(t) = w(t) - Cx(t). \tag{C.4}$$

The control law $u(t)$ is given by the following feedback controller$\int$

$$u(t) = K_{1i}x(t) + K_{2i}\varepsilon(t), \tag{C.5}$$

where K_{1i} and K_{2i} are the state feedback gains matrices to be synthesized. Then, the problem is reduced to determine optimal values of the controller gains.

The following Theorems summarize the solution to the fault diagnosis and the tracking controller

Theorem C.1.1. *(López-Estrada et al., 2014b) Given system (C.1), the observer (C.2) and let the attenuation level $\gamma > 0$. The residual state-space error system is globally stable with H_∞ performance if it satisfy $\| r(t) \|_2^2 \leq \gamma^2 \| d(t) \|_2^2$ and if there exist a matrix $Q = Q^T \geq 0$ and gain matrices $K_j = Q_1^{-1}\Phi_i$, $\forall i, j \in [1, 2, \ldots, h]$, such that:*

$$\begin{bmatrix} \mathsf{He}\left(A_i^T Q - C^T \Phi_i^T\right) & T_1 R_i & (WC)^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} \leq 0. \tag{C.6}$$

Theorem C.1.2. *(López-Estrada et al., 2014b) Given system (C.1), the comparator-integrator (C.2), the feedback controller defined by (C.5) and let the attenuation level $\gamma_c > 0$. The close loop system error system is globally stable with H_∞ performance if $\| x_c(t) \|_2^2 \leq \gamma_c^2 \| d(t) \|_2^2$ and if there exist a matrix $X = X^T \geq 0$ and gain matrices $K_j = X_1^{-1}\Xi_i$, $\forall i, j \in [1, 2, \ldots, h]$, such that:*

$$\begin{bmatrix} \mathsf{He}\left(X\bar{A}_{ci}^T + \Xi_i^T \bar{B}_{ci}^T\right) + \bar{R}_i \bar{R}_i^T & X \\ * & -\gamma^2 I \end{bmatrix} \leq 0. \tag{C.7}$$

All the proof and details can be consulted in the refereed paper.

This method was successfully applied to a Quadrotor UAV system, however the main drawback of the method is to considers measurable gain scheduling functions which reduce its applicability.

C.2 Discrete-time fault detection observer for discrete D-LPV systems

Consider the following discrete-time descriptor-linear parameter varying system affected by disturbances as:

$$\begin{aligned} Ex(k+1) &= \sum_{i=1}^{h} \rho_i(x(k)) \left[A_i x(k) + B_i u(k) + R_i d(k)\right] \\ y(k) &= Cx(k) + Hd(k) \end{aligned} \tag{C.8}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, $y \in \mathbb{R}^p$ and $d \in \mathbb{R}^l$ are the state vector, the control input and the measured output vector and the disturbances vector, respectively. A_i, B_i, R_i, C and, H are constant matrices of appropriate dimensions.

For an admissible and R/I-observable descriptor system (C.8), a robust discrete-time LPV observer is proposed as

$$z(k+1) = \sum_{i=1}^{h} \hat{\rho}_i \left[N_i z(k) + G_i u(k) + L_i y(k)\right]$$
$$\hat{x}(k) = z(k) + T_2 y(k) \tag{C.9}$$

where $z(k)$ is an auxiliary state vector, $\hat{x}(k)$ the state estimation of (C.8), N_i, G_i, L_i, T_2 are gain matrices of appropriate dimensions. An output error vector, also known as residual vector, is defined as:

$$r(k) = y(k) - \hat{y}(k) = y(k) - C\hat{x}(k) \tag{C.10}$$

The problem is reduced to find the gains matrices of the observer (C.9) such that $\lim_{k\to\infty} \|e(k)\| = \lim_{k\to\infty} \|x(k) - \hat{x}(k)\| \approx 0$; despite the disturbances and the error provided by the unmeasurable scheduling functions. Then by considering the perturbation method initially proposed for continuous systems, the system (C.8) is transformed as:

$$Ex(k+1) = \sum_{i=1}^{h} \hat{\rho}_i \left[A_i x(k) + B_i u(k) + R_i d(k)\right] + \omega(k)$$
$$y(k) = Cx(k) + Hd(k) \tag{C.11}$$

where the perturbation vector $\omega(k)$ has the following form:

$$\omega(k) = \sum_{i=1}^{h} (\rho_i - \hat{\rho}_i) \left[A_i x(k) + B_i u(k) + R_i d(k)\right]. \tag{C.12}$$

By considering the perturbed system (C.11) and the observer (C.9), the estimation error is given by

$$e(k) = x(k) - \hat{x}(k) \tag{C.13}$$
$$e(k) = (I - T_2 C)x(k) - z(k) - T_2 H d(k), \tag{C.14}$$

an its dynamics is obtained is written in the following augmented form:

$$e(k+1) = \sum_{i=1}^{h} \hat{\rho}_i \left[N_i e(k) + B_{\omega i} d_\omega(k)\right]$$
$$r(k) = Ce(k) + H_\omega d_w(k). \tag{C.15}$$

where

$$\begin{aligned} B_{\omega i} &= \begin{bmatrix} T_1R_i + K_iH - T_2H & T_1 \end{bmatrix}, d_\omega(k) = \begin{bmatrix} d(k) \\ \omega(k) \end{bmatrix}, \\ H_w &= \begin{bmatrix} H & 0 \end{bmatrix}. \end{aligned}$$

From (C.15) it is easy to deduce that the robust stability of (C.9) is achieved if the estimation error stabilize asymptotically to zero despite the perturbation vector $d_\omega(k)$. The following Theorem provides sufficient conditions to minimize the effects of $d_\omega(k)$ by considering the $\mathscr{L}_2$-gain from $d_\omega(k)$ to $r(k)$.

Theorem C.2.1. *(López-Estrada et al., 2014a) The state estimation error (C.15) between the D-LPV system (C.8) and the observer (C.9) is quadratically stable with H_∞ performance if there exist matrices $P = P^T \geq 0$, Ξ_i and scalar $\gamma > 0$ such that $\forall i \in [1, 2, ...h]$ the following condition hold:*

$$\begin{bmatrix} P & 0 & A_i^T T_1^T P + C^T \Xi_i^T & 0 & C^T \\ * & \gamma I & \Lambda_i & 0 & H^T \\ * & * & \gamma I & T_1^T P & 0 \\ * & * & * & P & 0 \\ * & * & * & * & I \end{bmatrix} \geq 0 \tag{C.16}$$

with $\Lambda_i = (T_1R_i)^T P + (\Xi_i H)^T - (T_2H)^T P$. The observer gain matrices are computed considering $K_i = P^{-1}\Xi_i$ and the conditions given in (C.18), (C.19) and

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} E \\ C \end{bmatrix}^\dagger \tag{C.17}$$

$$\begin{aligned} N_i &= T_1A_i + K_iC && \text{(C.18)} \\ K_i &= N_iT_2 - L_i. && \text{(C.19)} \end{aligned}$$

Proof details can be consulted in the cited paper. Note that Theorem C.2.1 gives sufficient conditions in the LMI formulation to compute the observer gains by considering $\mathscr{L}_2$-gain and a Lyapunov theory.

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